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Weyl and Riemann–Liouville multifractional Ornstein–Uhlenbeck processes

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Abstract

This paper considers two new multifractional stochastic processes, namely the Weyl multifractional Ornstein–Uhlenbeck process and the Riemann–Liouville multifractional Ornstein–Uhlenbeck process. Basic properties of these processes such as locally self-similar property and Hausdorff dimension are studied. The relationship between the multifractional Ornstein–Uhlenbeck processes and the corresponding multifractional Brownian motions is established.

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1. Introduction

Fractional Brownian motion [1] with its self-similarity property and stationary increments has been widely used as a model to describe phenomena with scaling and long-range dependence [2]. Its attractiveness lies in being simple as each of the properties is characterized and controlled by a single parameter H, the Hurst index. However, in real world data global self-similarity does not exist. Scaling only holds for a certain finite range of intervals or scales. In addition, empirical data indicate that the scaling exponent or order of self-similarity has more than one value. Thus a more realistic model requires a time-dependent (or space-dependent) scaling exponent. Multifractional Brownian motion [3, 4] with a variable Hurst index has been introduced precisely to address these problems. One decade after it was introduced, there exists an increasing interest in the study of multifractional Brownian motion from both theoretical and applied aspects. There exists various mathematical generalizations of multifractional Brownian motion such as its extension to generalized multifractional Brownian motion [5, 6], the generalization to allow the variable Hurst function to be a random function [7–9], its local time and small ball behaviour [10, 11], and its extension to n-dimensional multifractional Brownian field and sheet [6, 12, 13]. On the other hand, its practical applications stretch from

Internet traffic [14–16], financial time series [17, 18], biomedical and biomechanical systems [19–23] to laser propagation through turbulent media [24] and geomagnetic field studies [25].

Multifractional Brownian motion is useful in modelling long-range correlated phenomena with variable scaling property. There also exist many systems in nature which are short-range correlated and have variable memory [26–30]. Hence it would be useful to have a process which can play the role as the short-range dependent counterpart of multifractional Brownian motion. Based on the analogy of the relationship between fractional Brownian motion and the fractional Ornstein–Uhlenbeck process (also called fractional oscillator process), it would be natural to consider the possibility of generalizing the later to the multifractional Ornstein–Uhlenbeck (or multifractional oscillator) process as the counterpart of multifractional Brownian motion. Another physical motivation comes from the fact that the fractional Ornstein–Uhlenbeck process (the Weyl version) can be considered as a one-dimensional free Euclidean fractional Klein–Gordon field [31]. Such fractional derivative fields may play a role in the quantum field theory in fractal spacetime [32, 33]. Similarly, multifractional Ornstein–Uhlenbeck process can also play a similar role in quantum theory in multifractional spacetime. Recently, we have also witnessed a surge of interest in fractional dynamics [34–36] as models for various fractal transport phenomena in complex disordered media. There exist evidences that in certain physical systems, the fractal dimensions are dependent on some physical parameters such as time, position, conductance, etc [36–39]. Such processes can have variable memory, hence the need to use random processes which have variable local Hölder exponents. In view of the above reasons, we feel it is necessary to introduce a short-range dependent multifractional process which can be a candidate for modelling a wide range of physical systems with variable fractal dimension and short memory.

2. Weyl and Riemann–Liouville multifractional Ornstein–Uhlenbeck processes

2.1. Weyl and Riemann–Liouville fractional Ornstein–Uhlenbeck process

First we recall the definition of the Weyl Ornstein–Uhlenbeck process $Y_\alpha^W(t)$ and the Riemann–Liouville Ornstein–Uhlenbeck process $Y_\alpha^{RL}(t)$ [40]. They are processes defined respectively by stochastic integrals

$$Y_\alpha^W(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^t (t-u)^{\alpha-1}\,\mathrm{e}^{-\omega(t-u)}\,\mathrm{d}B(u),$$

$$Y_\alpha^{RL}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\,\mathrm{e}^{-\omega(t-u)}\,\mathrm{d}B(u),$$

where $\{B(u) : u \in \mathbb{R}\}$ is the standard Brownian motion and ω is a positive constant. To make sure that the processes $Y_\alpha^W(t)$ and $Y_\alpha^{RL}(t)$ have finite variance, we need to impose the condition $\alpha > 1/2$. These processes are obtained as solutions to the Weyl ($a = -\infty$) or Riemann–Liouville ($a = 0$) nonlinear fractional stochastic differential equation

$$({}_aD_t + \omega)^\alpha Y(t) = W(t), \tag{2.1}$$

known as the fractional Langevin equation. Here $W(t)$ is the standard white noise (formally the distributional derivative of the standard Brownian motion) and

$$({}_aD_t + \omega)^\alpha = \left(\mathrm{e}_a^{-\omega t}\,D_t\,\mathrm{e}^{\omega t}\right)^\alpha = \mathrm{e}_a^{-\omega t}\,D_t^\alpha\,\mathrm{e}^{\omega t},$$

where for $0 < \lambda < 1$, the fractional derivative ${}_aD_t^{-\lambda} = (\mathrm{d}/\mathrm{d}t)^{-\lambda}$ is defined by the integral

$${}_aD_t^{-\lambda} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\lambda-1} f(u)\,\mathrm{d}u, \qquad t \geqslant a,$$

and for $n-1 \leqslant \lambda < n$, ${}_aD_t^\lambda = (\mathrm{d}/\mathrm{d}t)^n\,{}_aD_t^{\lambda-n}$.

The Weyl fractional Ornstein–Uhlenbeck process $Y_\alpha^W(t)$ turns out to be a stationary centred Gaussian process with variance and covariance

$$E\left(\left[Y_\alpha^W(t)\right]^2\right) = \frac{\Gamma(2\alpha-1)(2\omega)^{1-2\alpha}}{\Gamma(\alpha)^2},$$
$$E\left(\left[Y_\alpha^W(t)Y_\alpha^W(s)\right]\right) = \frac{1}{\sqrt{\pi}\Gamma(\alpha)}\left(\frac{|t-s|}{2\omega}\right)^{\alpha-1/2} K_{\alpha-1/2}(\omega|t-s|), \qquad t \neq s, \tag{2.2}$$

where $K_\nu(z)$ is the modified Bessel function of second kind [41]. On the other hand, the Riemann–Liouville fractional Ornstein–Uhlenbeck process $Y_\alpha^{RL}(t)$ is a non-stationary centred Gaussian process with variance and covariance

$$E\left(\left[Y_\alpha^{RL}(t)\right]^2\right) = \frac{(2\omega)^{1-2\alpha}\gamma(2\alpha-1, 2\omega t)}{\Gamma(\alpha)^2},$$
$$E\left(\left[Y_\alpha^{RL}(t)Y_\alpha^{RL}(s)\right]\right) = \frac{\mathrm{e}^{-\omega(t+s)}s^\alpha t^{\alpha-1}}{\Gamma(\alpha+1)\Gamma(\alpha)}\Phi_1(1, 1-\alpha, 1+\alpha, s/t, 2\omega s), \qquad t > s,$$

where $\gamma(a, x)$ is the incomplete Gamma function, and $\Phi_1(a, b, c, x, y)$ is the confluent hypergeometric function in two variables [41].

When $\nu \in (0, 1)$, the function $K_\nu(z)$ has a series expansion (see 8.485 and 8.445 of [41])

$$K_\nu(z) = \frac{\pi}{2\sin(\pi\nu)}\sum_{m=0}^\infty\left(\frac{(z/2)^{2m-\nu}}{m!\Gamma(m+1-\nu)} - \frac{(z/2)^{2m+\nu}}{m!\Gamma(m+1+\nu)}\right)$$
$$= \frac{\pi}{2\sin(\pi\nu)}\left(\frac{(z/2)^{-\nu}}{\Gamma(1-\nu)} - \frac{(z/2)^\nu}{\Gamma(1+\nu)}\right) + O(z^{2-\nu}) \qquad \text{as} \quad z \to 0.$$

Using the formulae

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \qquad \Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}2^{1-2z}\Gamma(2z), \tag{2.3}$$

we find that

$$E\left(\left[Y_\alpha^W(t)Y_\alpha^W(s)\right]\right) = \frac{(2\omega)^{1-2\alpha}\Gamma(2\alpha-1)}{\Gamma(\alpha)^2} + \frac{|t-s|^{2\alpha-1}}{2\Gamma(2\alpha)\cos\pi\alpha} + \omega^{3-2\alpha}|t-s|^2 S_\alpha(\omega|t-s|), \tag{2.4}$$

where $S_\alpha(x)$ is a regular continuous function of x given explicitly by

$$S_\alpha(x) = -\frac{\sqrt{\pi}}{8\Gamma(\alpha)\cos(\pi\alpha)}\left[\sum_{m=0}^\infty \frac{x^{2m}}{2^{2m}(m+1)!\Gamma\left(m+\frac{5}{2}-\alpha\right)} - \left(\frac{x}{2}\right)^{2\alpha-1}\sum_{m=0}^\infty \frac{x^{2m}}{2^{2m}(m+1)!\Gamma\left(m+\frac{3}{2}+\alpha\right)}\right].$$

From (2.2) and (2.4), we see that in the limit $\omega \to 0$, both $E\left(\left[Y_\alpha^W(t)\right]^2\right)$ and $E\left(\left[Y_\alpha^W(t)Y_\alpha^W(s)\right]\right)$ are divergent. This explains why we cannot define the Weyl fractional Brownian motion as

$$\frac{1}{\Gamma(\alpha)}\int_{-\infty}^t (t-u)^{\alpha-1}\,\mathrm{d}B(u).$$

However, if we consider the normalized process (or we call it the reduced process analogous to the term used by Mandelbrot for fractional Brownian motion [1]) $X_\alpha^W(t) = Y_\alpha^W(t) - Y_\alpha^W(0)$, its covariance is

$$E\left(\left[X_\alpha^W(t)X_\alpha^W(s)\right]\right) = E\left(\left[Y_\alpha^W(t)Y_\alpha^W(s)\right]\right) + E\left(\left[Y_\alpha^W(0)\right]^2\right) - E\left(\left[Y_\alpha^W(t)Y_\alpha^W(0)\right]\right) - E\left(\left[Y_\alpha^W(s)Y_\alpha^W(0)\right]\right).$$

Using (2.4), we find that

$$E\big(\big[X_\alpha^W(t)X_\alpha^W(s)\big]\big) = -\frac{1}{2\cos(\pi\alpha)\Gamma(2\alpha)}(|t|^{2\alpha-1}+|s|^{2\alpha-1}-|t-s|^{2\alpha-1})$$
$$+\omega^{3-2\alpha}(|t-s|^2S_\alpha(\omega|t-s|)-|t|^2S_\alpha(\omega|t|)-|s|^2S_\alpha(\omega|s|)).$$

Therefore, when $1/2<\alpha<3/2$, as $\omega\to 0$, we have

$$E\big(\big[X_\alpha^W(t)X_\alpha^W(s)\big]\big) \overset{\omega\to 0}{\longrightarrow} -\frac{1}{2\cos(\pi\alpha)\Gamma(2\alpha)}(|t|^{2\alpha-1}+|s|^{2\alpha-1}-|t-s|^{2\alpha-1}).$$

Using the formulae in (2.3) and the formula $\Gamma(z+1)=z\Gamma(z)$, we can verify that by setting $H=\alpha-1/2$,

$$-\frac{1}{2\cos(\pi\alpha)\Gamma(2\alpha)} = \frac{\Gamma(1-2H)\cos(\pi H)}{2\pi H}. \tag{2.5}$$

Therefore, in the limit $\omega\to 0$, the reduced process $X_\alpha^W(t)$ approaches the reduced fractional Brownian motion $B_H(t)$ of Mandelbrot with starting value $B_H(0)=0$ [1], when we identify α with $H+1/2$. We show graphically in figures 1 and 2 how this limit process takes place. The $\omega\to 0$ limit of the Riemann–Liouville process $Y_\alpha^{RL}(t)$ will be discussed together with the multifractional case in section 2.3.

It is easy to see that when $H>0$, the expression in (2.5) is positive if and only if $H\in(2k,2k+1), k\in N$. Therefore, we need to impose the restriction $H<1$ for fractional Brownian motion so that its variance is nonnegative. However, both the Weyl and Riemann–Liouville fractional Ornstein–Uhlenbeck processes can be defined for any $\alpha>1/2$.

In the following, we generalize the Weyl and Riemann–Liouville fractional Ornstein–Uhlenbeck processes to their multifractional counterparts, with α depending on t. These generalizations are done along the same line as the generalization from fractional Brownian motion to multifractional Brownian motion [3]. Throughout, we assume that $\alpha(t)$ satisfies $\alpha(t)>1/2$, and it is Hölder continuous with exponent $0<\beta\leqslant 1$, i.e. there exists a constant K such that

$$|\alpha(t)-\alpha(s)|\leqslant K|t-s|^\beta \qquad \forall s,t.$$

2.2. *Weyl multifractional Ornstein–Uhlenbeck process*

The Weyl multifractional Ornstein–Uhlenbeck process is the stochastic process defined by the stochastic integral

$$Y_{\alpha(t)}^W(t) = \frac{1}{\Gamma(\alpha(t))}\int_{-\infty}^t (t-u)^{\alpha(t)-1}\,\mathrm{e}^{-\omega(t-u)}\,\mathrm{d}B(u).$$

Unlike the fractional process, it is a nontrivial issue whether this multifractional process can be obtained as the solution of a fractional differential equation like (2.1). Some work has been done to investigate the possibility of defining fractional differential equation of variable order (see, e.g., [42–48]) but is still at infancy.

The variance of the Weyl multifractional Ornstein–Uhlenbeck process is

$$E\big(\big[Y_{\alpha(t)}^W(t)\big]^2\big) = \frac{1}{\Gamma(\alpha(t))^2}\int_0^\infty u^{2\alpha(t)-2}\,\mathrm{e}^{-2\omega u}\,\mathrm{d}u = \frac{(2\omega)^{1-2\alpha(t)}\Gamma(2\alpha(t)-1)}{\Gamma(\alpha(t))^2}, \tag{2.6}$$

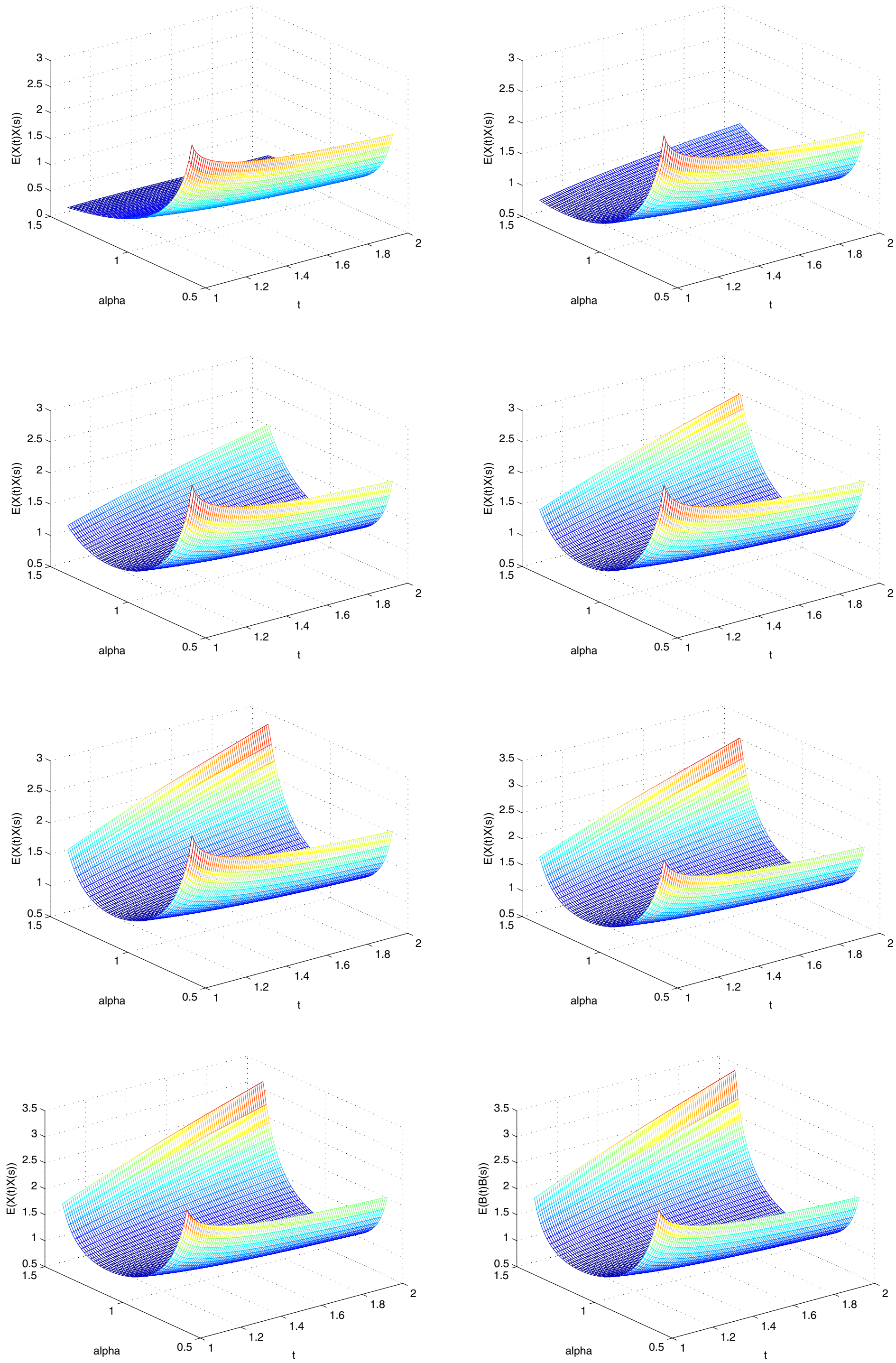

Figure 1. The first seven graphs are graphs for the correlation $E(X_\alpha^W(t)X_\alpha^W(s))$ when $\omega = 1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$ and $s = 1$. The last graph is the graph of the covariance of the fractional Brownian motion $B_{\alpha-(1/2)}(t)$.

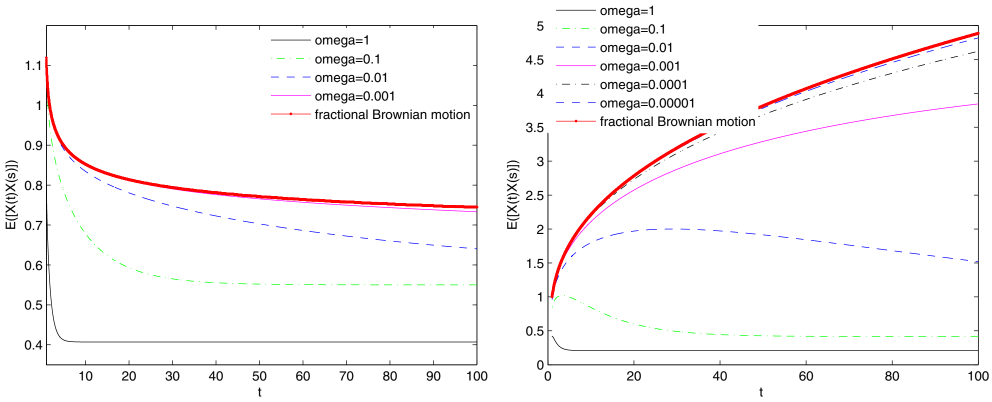

Figure 2. The graphs show how $E(X_\alpha^W(t)X_\alpha^W(s))$ approaches $E(B_{\alpha-\frac{1}{2}}(t)B_{\alpha-\frac{1}{2}}(s))$ when $\omega \to 0$ for $s = 1$, $\alpha = 0.9$ (left) and $\alpha = 1.2$ (right).

whereas for $s < t$, the covariance is given by (see 3.383 of [41])

$$\begin{aligned} E\big(Y_{\alpha(t)}^W(t)Y_{\alpha(s)}^W(s)\big) &= \frac{\mathrm{e}^{-\omega(t+s)}}{\Gamma(\alpha(t))\Gamma(\alpha(s))}\int_{-\infty}^s (t-u)^{\alpha(t)-1}(s-u)^{\alpha(s)-1}\,\mathrm{e}^{2\omega u}\,\mathrm{d}u \\ &= \frac{\mathrm{e}^{-\omega(t-s)}}{\Gamma(\alpha(t))\Gamma(\alpha(s))}\int_0^\infty u^{\alpha(s)-1}(u+t-s)^{\alpha(t)-1}\,\mathrm{e}^{-2\omega u}\,\mathrm{d}u \\ &= \frac{\mathrm{e}^{-\omega(t-s)}(t-s)^{\alpha(s)+\alpha(t)-1}}{\Gamma(\alpha(t))}\Psi(\alpha(s),\alpha(s)+\alpha(t),2\omega(t-s)), \end{aligned} \tag{2.7}$$

where $\Psi(\alpha,\gamma;z)$ is the confluent hypergeometric function. In contrast to the Weyl fractional Ornstein–Uhlenbeck process, the multifractional process is in general not stationary.

On the other hand, as in the fractional case, the variance and covariance functions are divergent as $\omega \to 0$. However, if we define $X_{\alpha(t)}^W(t) = Y_{\alpha(t)}^W(t) - Y_{\alpha(t)}^W(0)$, then we can show as in the fractional case that for $\alpha(t) \in (1/2, 3/2)$, by identifying $\alpha(t)$ with $H(t)+1/2$, in the limit $\omega \to 0$,

$$\begin{aligned} E\big(\big[X_{\alpha(t)}^W(t)X_{\alpha(s)}^W(s)\big]\big) \overset{\omega\to 0}{\longrightarrow} &-\frac{\Gamma(-H(t)-H(s))}{\pi}\bigg(\cos\Big(\frac{\pi}{2}(\Delta H_{t,s}+\operatorname{sign}(t)\bar{H}_{t,s})\Big)|t|^{H(t)+H(s)} \\ &+\cos\Big(\frac{\pi}{2}(\Delta H_{t,s}-\operatorname{sign}(s)\bar{H}_{t,s})\Big)|s|^{H(t)+H(s)} \\ &-\cos\Big(\frac{\pi}{2}(\Delta H_{t,s}+\operatorname{sign}(t-s)\bar{H}_{t,s})\Big)|t-s|^{H(t)+H(s)}\bigg), \end{aligned} \tag{2.8}$$

where $\Delta H_{t,s} = H(t) - H(s)$ and $\bar{H}_{t,s} = H(t) + H(s)$. The proof is complicated and we leave it to the appendix. By the result of [49], the right-hand side of (2.8) is the covariance of the multifractional Brownian motion $B_{H(t)}(t)$ defined by

$$\begin{aligned} B_{H(t)}(t) = \frac{1}{\Gamma(H(t)+1/2)}\bigg(&\int_{-\infty}^0((t-u)^{H(t)-1/2}-(-u)^{H(t)-1/2})\,\mathrm{d}u \\ &+\int_0^t(t-u)^{H(t)-1/2}\,\mathrm{d}u\bigg). \end{aligned} \tag{2.9}$$

In other words, the 'massless' limit of the reduced Weyl multifractional Ornstein–Uhlenbeck process is the multifractional Brownian motion (of moving average definition) introduced in [3].

For quite some time many authors believed that the multifractional Brownian motion introduced in [3] (the moving average definition) and [4] (the harmonizable representation) agree up to a multiplicative deterministic function. However in [49], Stoev and Taqqu showed that this is not the case. In this paper, when we say multifractional Brownian motion, it can be either of these definitions unless otherwise specified. For the multifractional Brownian motion introduced in [50], we will call it the Riemann–Liouville multifractional Brownian motion.

2.3. Riemann–Liouville multifractional Ornstein–Uhlenbeck process

The Riemann–Liouville multifractional Ornstein–Uhlenbeck process is the stochastic process defined by the stochastic integral

$$Y^{RL}_{\alpha(t)}(t) = \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-u)^{\alpha(t)-1}\,\mathrm{e}^{-\omega(t-u)}\,\mathrm{d}B(u).$$

Its variance is given by

$$E\big(\big[Y^{RL}_{\alpha(t)}(t)\big]^2\big) = \frac{1}{\Gamma(\alpha(t))^2}\int_0^t u^{2(\alpha(t)-1)}\,\mathrm{e}^{-2\omega u}\,\mathrm{d}u = \frac{(2\omega)^{1-2\alpha(t)}\gamma(2\alpha(t)-1, 2\omega t)}{\Gamma(\alpha(t))^2},$$

and for $0 < s < t$, its covariance is given by (see 3.385 of [41])

$$\begin{aligned} E\big(Y^{RL}_{\alpha(t)}(t)Y^{RL}_{\alpha(s)}(s)\big) &= \frac{\mathrm{e}^{-\omega(t+s)}}{\Gamma(\alpha(t))\Gamma(\alpha(s))}\int_0^s \mathrm{e}^{2\omega u}(t-u)^{\alpha(t)-1}(s-u)^{\alpha(s)-1}\,\mathrm{d}u \\ &= \frac{\mathrm{e}^{-\omega(t+s)}s^{\alpha(s)}t^{\alpha(t)-1}}{\Gamma(\alpha(t))\Gamma(\alpha(s))}\int_0^1 (1-u)^{\alpha(s)-1}\left(1-\frac{s}{t}u\right)^{\alpha(t)-1}\mathrm{e}^{2\omega us}\,\mathrm{d}u \\ &= \frac{\mathrm{e}^{-\omega(t+s)}s^{\alpha(s)}t^{\alpha(t)-1}}{\Gamma(\alpha(s)+1)\Gamma(\alpha(t))}\Phi_1(1, 1-\alpha(t), 1+\alpha(s), s/t, 2\omega s). \end{aligned} \tag{2.10}$$

Using the formulae in 8.354, 9.261 and 9.14 of [41], we have

$$\gamma(2\alpha(t)-1, 2\omega t) = \sum_{n=0}^{\infty}\frac{(-1)^n(2\omega t)^{2\alpha(t)+n-1}}{n!(2\alpha(t)+n-1)},$$

$$\begin{aligned} \Phi_1(1, 1-\alpha(t), 1+\alpha(s), s/t, 2\omega s) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!(1-\alpha(t))_m}{(1+\alpha(s))_{m+n}m!n!}\left(\frac{s}{t}\right)^m(2\omega s)^n \\ &= \sum_{n=0}^{\infty}\frac{(2\omega s)^n}{(1+\alpha(s))_n}\,{}_2F_1(1+n, 1-\alpha(t); 1+n+\alpha(s); s/t), \end{aligned}$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ is the Gauss hypergeometric function. Using this, we find that in the limit $\omega \to 0$,

$$\begin{aligned} E\big(\big[Y^{RL}_{\alpha(t)}(t)\big]^2\big) &\to \frac{t^{2\alpha(t)-1}}{(2\alpha(t)-1)\Gamma(\alpha(t))^2}, \\ E\big(Y^{RL}_{\alpha(t)}(t)Y^{RL}_{\alpha(s)}(s)\big) &\to \frac{s^{\alpha(s)}t^{\alpha(t)-1}}{\Gamma(\alpha(s)+1)\Gamma(\alpha(t))}\,{}_2F_1(1, 1-\alpha(t); 1+\alpha(s); s/t), \end{aligned} \tag{2.11}$$

which are the variance and the covariance of the Riemann–Liouville multifractional Brownian motion $X^{RL}_{H(t)}(t)$ [51] if we identify $\alpha(t)$ with $H(t)+1/2$, $H(t) > 0$ being the Hurst function of the multifractional process. Thus we can regard the Riemann–Liouville multifractional Brownian motion as the 'massless' or $\omega \to 0$ limit (in the sense of finite-dimensional distributions) of Riemann–Liouville multifractional Ornstein–Uhlenbeck process.

3. Some bounds on the variance of the increment processes

In order to study the local properties of the Weyl and Riemann–Liouville multifractional Ornstein–Uhlenbeck processes, we need to know the leading terms and some bounds of the variances of their increment processes. This section serves as a preparation for later sections. For simplicity, we only consider the case where $\alpha(t) < 3/2$ for all t. In fact, this is also the case where the local properties exhibit strong dependence on the value of $\alpha(t)$.

Given a compact interval $[a, b] \subseteq \mathbb{R}$, we let $m_\alpha[a, b] = \min\{\alpha(t) : t \in [a, b]\}$ and $M_\alpha[a, b] = \max\{\alpha(t) : t \in [a, b]\}$. By our assumption on $\alpha(t)$, $m_\alpha[a, b] > 1/2$ and $M_\alpha[a, b] < 3/2$.

First, for the Weyl case, we have:

Lemma 3.1. *Let $[a, b] \subseteq \mathbb{R}$ be a closed interval.*

A. There exists a constant C_1^W depending only on $[a, b]$ such that

$$E\left(\left[Y^W_{\alpha(t+\tau)}(t+\tau) - Y^W_{\alpha(t)}(t)\right]^2\right) \leqslant C_1^W |\tau|^{\min\{2m_\alpha[a,b]-1, 2\beta\}}$$

for all $t, t+\tau \in [a, b]$ satisfying $|\tau| < 1$.

B. If we make the further assumption that $\alpha(t) - 1/2 < \beta$ for all t, then

I. There exist constants C_2^W and $\delta < 1$ depending only on $[a, b]$ such that

$$E\left(\left[Y^W_{\alpha(t+\tau)}(t+\tau) - Y^W_{\alpha(t)}(t)\right]^2\right) \geqslant C_2^W |\tau|^{2M_\alpha[a,b]-1}$$

whenever $t, t+\tau \in [a, b]$ satisfying $|\tau| < \delta$.

II. $E\left(\left[Y^W_{\alpha(t+\tau)}(t+\tau) - Y^W_{\alpha(t)}(t)\right]^2\right) = -\dfrac{|\tau|^{2\alpha(t)-1}}{\Gamma(2\alpha(t))\cos(\pi\alpha(t))} + O(|\tau|^{\beta+\alpha(t)-1/2})$

as $\tau \to 0$.

Proof. For fixed t, we define

$$V_t^W(\tau) = Y^W_{\alpha(t+\tau)}(t+\tau) - Y^W_{\alpha(t)}(t+\tau).$$

Then we have

$$E\left(\left[Y^W_{\alpha(t+\tau)}(t+\tau) - Y^W_{\alpha(t)}(t)\right]^2\right) = E\left(\left[Y^W_{\alpha(t)}(t+\tau) - Y^W_{\alpha(t)}(t)\right]^2\right) + E\left(\left[V_t^W(\tau)\right]^2\right) + 2E\left(\left[Y^W_{\alpha(t)}(t+\tau) - Y^W_{\alpha(t)}(t)\right]\left[V_t^W(\tau)\right]\right). \tag{3.1}$$

Using (2.6) and (2.4), we have

$$E\left(\left[Y^W_{\alpha(t)}(t+\tau) - Y^W_{\alpha(t)}(t)\right]^2\right) = E\left(\left[Y^W_{\alpha(t)}(t+\tau)\right]^2\right) + E\left(\left[Y^W_{\alpha(t)}(t)\right]^2\right) - 2E\left(Y^W_{\alpha(t)}(t+\tau)Y^W_{\alpha(t)}(t)\right)$$
$$= -\frac{|\tau|^{2\alpha(t)-1}}{\Gamma(2\alpha(t))\cos\pi\alpha(t)} + |\tau|^2 R_{\alpha(t)}(|\tau|), \tag{3.2}$$

where $R_\lambda(x)$ is a continuous regular function of $x \in \mathbb{R}$ and $\lambda \in (0, 1)$. Let C_1 be the maximum value of $-1/(\Gamma(2\lambda)\cos(\pi\lambda))$ when $\lambda \in [m_\alpha[a, b], M_\alpha[a, b]]$ and let C_2 be the maximum value of $|R_\lambda(\eta)|$ when $\lambda \in [m_\alpha[a, b], M_\alpha[a, b]]$ and $|\eta| \leqslant 1$. Then when $|\tau| < 1$,

$$E\left(\left[Y^W_{\alpha(t)}(t+\tau) - Y^W_{\alpha(t)}(t)\right]^2\right) \leqslant C_1|\tau|^{2\alpha(t)-1} + C_2|\tau|^2 \leqslant (C_1 + C_2)|\tau|^{2m_\alpha[a,b]-1}.$$

On the other hand, we can write

$$V_t^W(\tau) = V^W_{t,1}(\tau) + V^W_{t,2}(\tau),$$

where

$$V_{t,1}^W(\tau)=\left(\frac{1}{\Gamma(\alpha(t+\tau))}-\frac{1}{\Gamma(\alpha(t))}\right)\int_{-\infty}^{t+\tau}(t+\tau-u)^{\alpha(t+\tau)-1}\,\mathrm{e}^{-\omega(t+\tau-u)}\,\mathrm{d}B(u),$$
$$V_{t,2}^W(\tau)=\frac{1}{\Gamma(\alpha(t))}\int_{-\infty}^{t+\tau}((t+\tau-u)^{\alpha(t+\tau)-1}-(t+\tau-u)^{\alpha(t)-1})\,\mathrm{e}^{-\omega(t+\tau-u)}\,\mathrm{d}B(u).$$

A straightforward computation gives

$$E\big(\big[V_{t,1}^W(\tau)\big]^2\big)=\left(\frac{1}{\Gamma(\alpha(t+\tau))}-\frac{1}{\Gamma(\alpha(t))}\right)^2(2\omega)^{1-2\alpha(t+\tau)}\Gamma(2\alpha(t+\tau)-1),$$
$$\begin{aligned}E\big(\big[V_{t,2}^W(\tau)\big]^2\big)=&\frac{1}{\Gamma(\alpha(t))^2}((2\omega)^{1-2\alpha(t+\tau)}\Gamma(2\alpha(t+\tau)-1)+(2\omega)^{1-2\alpha(t)}\Gamma(2\alpha(t)-1)\\&-2(2\omega)^{1-\alpha(t)-\alpha(t+\tau)}\Gamma(\alpha(t)+\alpha(t+\tau)-1)).\end{aligned}$$

As the function $z\mapsto g_1(z)=1/\Gamma(z)$ is analytic for $m_\alpha[a,b]\leqslant z\leqslant M_\alpha[a,b]$, by mean value theorem,

$$\frac{1}{\Gamma(\alpha(t+\tau))}-\frac{1}{\Gamma(\alpha(t))}=g_1'(\zeta_{t,\tau})(\alpha(t+\tau)-\alpha(t))$$

for some $\zeta_{t,\tau}\in(m_\alpha[a,b],M_\alpha[a,b])$. Since a continuous function has a maximum on any compact interval, and α is Hölder continuous, we have $E\big(\big[V_{t,1}^W(\tau)\big]^2\big)\leqslant C_3|\tau|^{2\beta}$ for some constant C_3 depending only on $[a,b]$.

Similarly, the function $z\mapsto g_2(z)=(2\omega)^{1-2z}\Gamma(2z-1)$ is analytic for $m_\alpha[a,b]\leqslant z\leqslant M_\alpha[a,b]$. Therefore, for any x,h so that $x,x+2h$ are in the interval $[m_\alpha[a,b],M_\alpha[a,b]]$,

$$|g_2(x+2h)+g_2(x)-2g_2(x+h)|=\left|\int_x^{x+h}(f'(y+h)-f'(y))\,\mathrm{d}y\right|\leqslant C_4|h|^2,$$

for some constant C_4 depending only on $[a,b]$. This gives $E\big(\big[V_{t,2}^W(\tau)\big]^2\big)\leqslant C_5|\tau|^{2\beta}$ for some constant C_5 depending only on $[a,b]$.

Therefore,

$$E\big(\big[V_t^W(\tau)\big]^2\big)\leqslant 2\big(E\big(\big[V_{t,1}^W(\tau)\big]^2\big)+E\big(\big[V_{t,2}^W(\tau)\big]^2\big)\big)\leqslant C_6|\tau|^{2\beta},\tag{3.3}$$

where $C_6=2(C_3+C_5)$. By Cauchy–Schwarz inequality,

$$\begin{aligned}\big|E\big(\big[Y_{\alpha(t)}^W(t+\tau)-Y_{\alpha(t)}^W(t)\big]\big[V_t^W(\tau)\big]\big)\big|&\leqslant E\big(\big[Y_{\alpha(t)}^W(t+\tau)-Y_{\alpha(t)}^W(t)\big]^2\big)^{1/2}E\big(\big[V_t^W(\tau)\big]^2\big)^{1/2}\\&\leqslant C_7|\tau|^{\alpha(t)-1/2+\beta}\leqslant C_7|\tau|^{m_\alpha[a,b]-1/2+\beta},\quad\text{where}\quad C_7=\sqrt{(C_1+C_2)C_6}.\end{aligned}\tag{3.4}$$

A follows immediately. On the other hand, by assuming that $\beta>M_\alpha[a,b]-1/2\geqslant\alpha(t)-1/2$, we immediately obtain II of B from (3.1)–(3.4). Using these same equations again, we obtain

$$\begin{aligned}E\big(\big[Y_{\alpha(t+\tau)}^W(t+\tau)-Y_{\alpha(t)}^W(t)\big]^2\big)&\geqslant C_8|\tau|^{2\alpha(t)-1}-C_2|\tau|^2-C_6|\tau|^{2\beta}-2C_7|\tau|^{\alpha(t)-1/2+\beta}\\&=|\tau|^{2\alpha(t)-1}(C_8-C_2|\tau|^{3-2\alpha(t)}-C_6|\tau|^{2\beta-2\alpha(t)+1}-2C_7|\tau|^{\beta-\alpha(t)+1/2})\\&\geqslant|\tau|^{2M_\alpha[a,b]-1}(C_8-C_2|\tau|^{3-2M_\alpha[a,b]}-C_6|\tau|^{2\beta-2M_\alpha[a,b]+1}-2C_7|\tau|^{\beta-M_\alpha[a,b]+1/2}),\end{aligned}$$

where C_8 is the minimum value of $-1/(\Gamma(2\lambda)\cos(\pi\lambda))$ when $\lambda\in[m_\alpha[a,b],M_\alpha[a,b]]$. Since

$$C_8-C_2|\tau|^{3-2M_\alpha[a,b]}-C_6|\tau|^{2\beta-2M_\alpha[a,b]+1}-2C_7|\tau|^{\beta-M_\alpha[a,b]+1/2}$$

is continuous in τ and approaches C_8 when $\tau\to 0$, there exists $\delta>0$ such that for $|\tau|<\delta$, it is greater than $C_8/2$. This gives I of B. □

By using analogous arguments, we can prove a result similar to lemma 3.1 for the Riemann–Liouville process.

Lemma 3.2. *Let* $[a, b]$ *be a closed interval in* $\mathbb{R}^+$,

A. There exists a constant C_1^{RL} *depending only on* $[a, b]$ *such that*

$$E\left(\left[Y^{RL}_{\alpha(t+\tau)}(t+\tau) - Y^{RL}_{\alpha(t)}(t)\right]^2\right) \leqslant C_1^{RL}|\tau|^{\min\{2m_\alpha - 1, 2\beta\}}$$

for all $t, t+\tau \in [a, b]$ *satisfying* $|\tau| < 1$.

B. If we make the further assumption that $\alpha(t) - 1/2 < \beta$ *for all* t, *then*

I. There exist constants C_2^{RL} *and* $\delta < 1$ *depending only on* $[a, b]$ *such that*

$$E\left(\left[Y^{RL}_{\alpha(t+\tau)}(t+\tau) - Y^{RL}_{\alpha(t)}(t)\right]^2\right) \geqslant C_2^{RL}|\tau|^{2M_\alpha - 1}$$

whenever $t, t+\tau \in [a, b]$ *are such that* $|\tau| < \delta$.

II. $E\left(\left[Y^{RL}_{\alpha(t+\tau)}(t+\tau) - Y^{RL}_{\alpha(t)}(t)\right]^2\right) = -\dfrac{|\tau|^{2\alpha(t)-1}}{\Gamma(2\alpha(t))\cos\pi\alpha(t)} + O(|\tau|^{\alpha(t)-1/2+\beta})$

as $\tau \to 0$.

Proof. Let

$$Q_{\alpha(t)}(t) = Y^W_{\alpha(t)}(t) - Y^{RL}_{\alpha(t)}(t) = \frac{1}{\Gamma(\alpha(t))}\int_{-\infty}^0 (t-u)^{2\alpha(t)-1}\,\mathrm{e}^{-\omega(t-u)}\,\mathrm{d}B(u).$$

For any t and s, $Y^{RL}_{\alpha(t)}(t)$ and $Q_{\alpha(s)}(s)$ are independent random variables. Therefore

$$E\left(\left[Y^{RL}_{\alpha(t+\tau)}(t+\tau) - Y^{RL}_{\alpha(t)}(t)\right]^2\right) = E\left(\left[Y^{W}_{\alpha(t+\tau)}(t+\tau) - Y^{W}_{\alpha(t)}(t)\right]^2\right) - E\left(\left[Q_{\alpha(t+\tau)}(t+\tau) - Q_{\alpha(t)}(t)\right]^2\right).$$

Without loss of generality, we assume that $\tau \geqslant 0$ and write

$$Q_{\alpha(t+\tau)}(t+\tau) - Q_{\alpha(t)}(t) = Q_1(t,\tau) + Q_2(t,\tau) + Q_3(t,\tau) + Q_4(t,\tau),$$

where

$$Q_1(t,\tau) = \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right)\int_{-\infty}^0 (t+\tau-u)^{\alpha(t+\tau)-1}\,\mathrm{e}^{-\omega(t+\tau-u)}\,\mathrm{d}B(u),$$

$$Q_2(t,\tau) = \frac{1}{\Gamma(\alpha(t))}\int_{-\infty}^0 (t+\tau-u)^{\alpha(t+\tau)-1}(\mathrm{e}^{-\omega(t+\tau-u)} - \mathrm{e}^{-\omega(t-u)})\,\mathrm{d}B(u),$$

$$Q_3(t,\tau) = \frac{1}{\Gamma(\alpha(t))}\int_{-\infty}^0 ((t-u)^{\alpha(t+\tau)-1} - (t-u)^{\alpha(t)-1})\,\mathrm{e}^{-\omega(t-u)}\,\mathrm{d}B(u),$$

$$Q_4(t,\tau) = \frac{1}{\Gamma(\alpha(t))}\int_{-\infty}^0 ((t+\tau-u)^{\alpha(t+\tau)-1} - (t-u)^{\alpha(t+\tau)-1})\,\mathrm{e}^{-\omega(t-u)}\,\mathrm{d}B(u).$$

As in the proof of lemma 3.1, it is easy to see that whenever $t, t+\tau \in [a, b]$ is such that $|\tau| < 1$,

$$E([Q_1(t,\tau)]^2) = \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right)^2 (2\omega)^{1-2\alpha(t+\tau)}\Gamma(2\alpha(t+\tau)-1, 2\omega(t+\tau)) \leqslant C_1|\tau|^{2\beta},$$

$$E([Q_2(t,\tau)]^2) = \frac{(\mathrm{e}^{\omega\tau}-1)^2}{\Gamma(\alpha(t))^2}(2\omega)^{1-2\alpha(t+\tau)}\Gamma(2\alpha(t+\tau)-1, 2\omega(t+\tau)) \leqslant C_2|\tau|^2,$$

$$E([Q_3(t,\tau)]^2) = \frac{1}{\Gamma(\alpha(t))^2}\Big((2\omega)^{1-2\alpha(t+\tau)}\Gamma(2\alpha(t+\tau)-1, 2\omega t) + (2\omega)^{1-2\alpha(t)}\Gamma(2\alpha(t)-1, 2\omega t) - (2\omega)^{1-\alpha(t)-\alpha(t+\tau)}\Gamma(\alpha(t)+\alpha(t+\tau)-1, 2\omega t)\Big) \leqslant C_3|\tau|^{2\beta}$$

for some constants C_1, C_2, C_3 that depend only on $[a,b]$. On the other hand,

$$E([Q_4(t,\tau)]^2) = \frac{1}{\Gamma(\alpha(t))^2}\int_t^\infty ((u+\tau)^{\alpha(t+\tau)-1} - u^{\alpha(t+\tau)-1})^2 \,\mathrm{e}^{-2\omega u}\,\mathrm{d}u$$

and by mean value theorem, there exists $v(u) \in (u, u+\tau)$ such that

$$(u+\tau)^{\alpha(t+\tau)-1} - u^{\alpha(t+\tau)-1} = (\alpha(t+\tau)-1)\tau v(u)^{\alpha(t+\tau)-2}.$$

Since $\alpha(t+\tau) < 2$, $v(u)^{\alpha(t+\tau)-2} \leqslant u^{\alpha(t+\tau)-2}$, we find that

$$E([Q_4(t,\tau)]^2) \leqslant (\alpha(t+\tau)-1)^2\tau^2 \int_t^\infty u^{2\alpha(t+\tau)-4}\,\mathrm{e}^{-2\omega u}\,\mathrm{d}u \leqslant C_4|\tau|^2,$$

where C_4 depends only on $[a,b]$. By Cauchy–Schwarz inequality and the fact that $|\tau|^2 \leqslant |\tau|^{2\beta}$ for $|\tau| < 1$, we conclude that

$$E([Q_{\alpha(t+\tau)}(\tau) - Q_{\alpha(t)}(t)]^2) \leqslant 4\sum_{i=1}^{4} E([Q_i(t,\tau)]^2) \leqslant C_5|\tau|^{2\beta},$$

with $C_5 = 4(C_1+C_2+C_3+C_4)$. The assertions now follow as in lemma 3.1. □

The following result can be obtained from A of lemma 3.1 and lemma 3.2 based on standard argument.

Proposition 3.3. *With probability one, the paths $Y^W_{\alpha(t)}(t)$ and $Y^{RL}_{\alpha(t)}(t)$ are continuous.*

Proof. First we consider the Weyl process. Since continuity is a local property, it is sufficient to prove continuity of $Y^W_{\alpha(t)}(t)$ in any interval $[a,b]$ satisfying $b-a<1$. Given $r>0$ and $t, t+\tau \in [a,b]$, since $Y^W_{\alpha(t+\tau)}(t+\tau) - Y^W_{\alpha(t)}(t)$ is a normal distribution, there exists a constant B_r depending only on r such that

$$E\big(\big|Y^W_{\alpha(t+\tau)}(t+\tau) - Y^W_{\alpha(t)}(t)\big|^r\big) = B_r E\big(\big[Y^W_{\alpha(t+\tau)}(t+\tau) - Y^W_{\alpha(t)}(t)\big]^2\big)^{r/2} \leqslant B_r\big(C_1^W\big)^{r/2}|\tau|^{r\min\{m_\alpha[a,b]-1/2,\beta\}}.$$

Choose r so that $r\min\{m_\alpha[a,b]-1/2,\beta\} > 1$, Kolmogorov's continuity theorem implies that $Y^W_{\alpha(t)}(t)$ is continuous on $[a,b]$. The Riemann–Liouville process is proved analogously. □

4. Hölder exponents

In this section and the followings, we would like to study the local properties of the multifractional Ornstein–Uhlenbeck processes such as their Hölder exponents, the Hausdorff and box dimensions of their graphs and the local asymptotic self-similarity. All these properties are closely related. From now on, we make the further assumption that $\alpha(t) - 1/2 < \beta$ for all t in the domain.

Proposition 4.1. *We have the followings:*

A. Given an interval $[a, b] \subseteq \mathbb{R}$ *and any* $0 \leqslant \lambda < m_\alpha[a, b] - 1/2$*, with probability one, there exists a constant* $c_{1,\lambda}$ *depending only on* $[a, b]$ *and* λ *such that*

$$\left|Y^W_{\alpha(t)}(t) - Y^W_{\alpha(s)}(s)\right| \leqslant c_{1,\lambda}|t - s|^\lambda \qquad \textit{whenever} \quad t, s \in [a, b].$$

B. Given an interval $[a, b] \subseteq \mathbb{R}^+$ *and any* $0 \leqslant \lambda < m_\alpha[a, b] - 1/2$*, with probability one, there exists a constant* $c_{2,\lambda}$ *depending only on* $[a, b]$ *and* λ *such that*

$$\left|Y^{RL}_{\alpha(t)}(t) - Y^{RL}_{\alpha(s)}(s)\right| \leqslant c_{2,\lambda}|t - s|^\lambda \qquad \textit{whenever} \quad t, s \in [a, b].$$

Proof. First, let $t, t + \tau \in [a, b]$ with $0 < \tau < 1$. By Markov inequality, for any $r \geqslant 0$,

$$P\left(\left|Y^W_{\alpha(t+\tau)}(t+\tau) - Y^W_{\alpha(t)}(t)\right| > \tau^\lambda\right) \leqslant \frac{1}{\tau^{r\lambda}} E\left(\left|Y^W_{\alpha(t+\tau)}(t+\tau) - Y^W_{\alpha(t)}(t)\right|^r\right)$$
$$\leqslant \frac{B_r\left(C^W_1\right)^{r/2}\tau^{r(m_\alpha[a,b]-1/2)}}{\tau^{r\lambda}}.$$

Choose r such that

$$r(m_\alpha[a, b] - 1/2 - \lambda) = 2.$$

Then

$$P\left(\left|Y^W_{\alpha(t+\tau)}(t+\tau) - Y^W_{\alpha(t)}(t)\right| > \tau^\lambda\right) \leqslant C\tau^2,$$

where $C = B_r\left(C^W_1\right)^{r/2}$ is a constant that depends only on $[a, b]$. Given a positive integer k, let $t_{j,k} = a + 2^{-k} j$ for $0 \leqslant j \leqslant 2^k(b - a)$, and let A_n be the event

$$\left|Y^W_{\alpha(t_{j,k})}(t_{j,k}) - Y^W_{\alpha(t_{j-1,k})}(t_{j-1,k})\right| > 2^{-k\lambda} \qquad \text{for some} \quad k \geqslant n \text{ and } 1 \leqslant j \leqslant 2^k(b - a).$$

Let m be the smallest positive integer so that $m \geqslant b - a$. Then

$$P(A_n) \leqslant Cm\sum_{k=n}^\infty 2^k 2^{-2k} = Cm2^{-n+1}.$$

It easily follows that $\sum_{n=1}^\infty P(A_n) < \infty$. Therefore by Borel–Cantelli lemma, $P(\sup_n A_n) = 0$, i.e., with probability one, there exists an integer N (depending only on $[a, b]$ and λ) such that

$$\left|Y^W_{\alpha(t_{j,k})}(t_{j,k}) - Y^W_{\alpha(t_{j-1,k})}(t_{j-1,k})\right| \leqslant 2^{-k\lambda} \qquad \text{for all} \quad k \geqslant N \text{ and all } 1 \leqslant j \leqslant 2^k(b - a).$$

Given $s < t \in [a, b]$ with $t - s \leqslant 2^{-N}$, let $u \geqslant N$ be the smallest positive integer such that $2^{-u} \leqslant t - s$. We can write the interval $[s, t]$ as a union of intervals of the form $[a + (j - 1)2^{-k}, a + j2^{-k})]$, with $k \geqslant u \geqslant K$ and at most two j for every k. Then by triangle inequality, with probability one,

$$\left|Y^W_{\alpha(t)}(t) - Y^W_{\alpha(s)}(s)\right| \leqslant 2\sum_{k=u}^\infty 2^{-k\lambda} \leqslant \frac{22^{-\lambda u}}{1 - 2^{-\lambda}} \leqslant \frac{2}{1 - 2^{-\lambda}}|t - s|^\lambda.$$

Now for arbitrary $s < t \in [a, b]$, we can find $q - 1$ points $s_1, s_2, \ldots, s_{q-1} \in [a, b]$ with $q \leqslant (b - a)2^N + 1$ and $|s_i - s_{i-1}| < 2^{-N}$ for $1 \leqslant i \leqslant q$, where $s_0 = s$ and $s_q = t$. Triangle inequality then gives

$$\left|Y^W_{\alpha(t)}(t) - Y^W_{\alpha(s)}(s)\right| \leqslant \frac{2q}{1 - 2^{-\lambda}}|t - s|^\lambda \leqslant \frac{2[(b - a)2^N + 1]}{1 - 2^{-\lambda}}|t - s|^\lambda.$$

The Riemann–Liouville case is proved in the same way. □

Recall that a real-valued function f is said to have Hölder exponent β at a point t_0 if and only if

$$\lim_{h\to 0}\frac{|f(t_0+h)-f(t_0)|}{|h|^{\lambda}}=0 \qquad \text{for any} \quad \lambda<\beta, \text{ and}$$
$$\limsup_{h\to 0}\frac{|f(t_0+h)-f(t_0)|}{|h|^{\lambda}}=\infty \qquad \text{for any} \quad \lambda>\beta.$$

Similar to the multifractional Brownian motion, we have

Proposition 4.2. *With probability one, both the functions $Y^{W}_{\alpha(t)}(t)$ and $Y^{RL}_{\alpha(t)}(t)$ have Hölder exponent $\alpha(t_0)-1/2$ at the point t_0 in the domain.*

Proof. Given $\lambda<\alpha(t_0)-1/2$, by continuity of α, there exists $\delta>0$ such that $\alpha(t)-1/2>\lambda$ for all t in the interval $[t_0-\delta,t_0+\delta]$. This implies that $m_\alpha[t_0-\delta,t_0+\delta]-1/2>\lambda$. Let $\lambda'=1/2(m_\alpha[t_0-\delta,t_0+\delta]-1/2+\lambda)$. Then $\lambda<\lambda'<m_\alpha[t_0-\delta,t_0+\delta]-1/2$. It follows from proposition 4.1 that with probability one,

$$\frac{\left|Y^{W}_{\alpha(t_0+h)}(t_0+h)-Y^{W}_{\alpha(t_0)}(t_0)\right|}{|h|^{\lambda}}\leqslant c_{1,\lambda'}|h|^{\lambda'-\lambda} \qquad \text{for all} \quad |h|<\delta.$$

Therefore,

$$\lim_{h\to 0}\frac{\left|Y^{W}_{\alpha(t_0+h)}(t_0+h)-Y^{W}_{\alpha(t_0)}(t_0)\right|}{|h|^{\lambda}}=0 \qquad \text{with probability one.}$$

On the other hand, given $\lambda>\alpha-1/2$, let $\delta_1>0$ be as given by B.I. of lemma 3.1 and let $\delta_2>0$ be such that $\alpha(t)-1/2<\lambda$ for all $t\in[t_0-\delta_2,t_0+\delta_2]$. Define $\delta=\min\{\delta_1,\delta_2\}$ and let h_n be a sequence so that $|h_n|<\delta$ and $h_n\to 0$ as $n\to\infty$. Define $\sigma_{t_0,h}$ as the nonnegative number so that

$$\sigma^2_{t_0,h}=E\big(\big[Y^{W}_{\alpha(t_0+h)}(t_0+h)-Y^{W}_{\alpha(t_0)}(t_0)\big]^2\big).$$

Then by B.I. of lemma 3.1 and the fact that $\big(Y^{W}_{\alpha(t_0+h)}(t_0+h)-Y^{W}_{\alpha(t_0)}(t_0)\big)\big/\sigma_{t_0,h}$ is the standard normal distribution,

$$\begin{aligned}
P\left(\left|\frac{h_n^{\lambda}}{Y^{W}_{\alpha(t_0+h_n)}(t_0+h_n)-Y^{W}_{\alpha(t_0)}(t_0)}\right|>\varepsilon\right)&=P\left(\left|\frac{Y^{W}_{\alpha(t_0+h_n)}(t_0+h_n)-Y^{W}_{\alpha(t_0)}(t_0)}{\sigma_{t_0,h_n}}\right|<\frac{|h_n|^{\lambda}}{\varepsilon\sigma_{t_0,h_n}}\right)\\
&\leqslant\big(C_2^W\big)^{-1/2}P\left(\left|\frac{Y^{W}_{\alpha(t_0+h_n)}(t_0+h_n)-Y^{W}_{\alpha(t_0)}(t_0)}{\sigma_{t_0,h_n}}\right|<\varepsilon^{-1}|h_n|^{\lambda-M_\alpha[t_0-\delta,t_0+\delta]+1/2}\right)\\
&=\frac{2\big(C_2^W\big)^{-1/2}}{\sqrt{2\pi}}\int_0^{\varepsilon^{-1}|h_n|^{\lambda-M_\alpha[t_0-\delta,t_0+\delta]+1/2}}\mathrm{e}^{-u^2/2}\,\mathrm{d}u\\
&\leqslant\frac{2\big(C_2^W\big)^{-1/2}}{\sqrt{2\pi}}\frac{|h_n|^{\lambda-M_\alpha[t_0-\delta,t_0+\delta]+1/2}}{\varepsilon}.
\end{aligned}$$

We can find a subsequence h_{n_j} such that $2h_{n_{j+1}}<h_{n_j}$ for all j. Since $\lambda>M_\alpha[t_0-\delta,t_0+\delta]-1/2$, we have

$$\sum_j P\left(\left|\frac{h_{n_j}^{\lambda}}{Y^{W}_{\alpha(t_0+h_{n_j})}\big(t_0+h_{n_j}\big)-Y^{W}_{\alpha(t_0)}(t_0)}\right|>\varepsilon\right)<\infty.$$

By Borel–Cantelli lemma, this implies that

$$P\left(\left|\frac{h_{n_j}^{\lambda}}{Y^{W}_{\alpha(t_0+h_{n_j})}\big(t_0+h_{n_j}\big)-Y^{W}_{\alpha(t_0)}(t_0)}\right|>\varepsilon\ \text{ i.o.}\right)=0.$$

Therefore

$$\lim_{j\to\infty} \frac{h_{n_j}^{\lambda}}{Y^W_{\alpha(t_0+h_{n_j})}\big(t_0+h_{n_j}\big) - Y^W_{\alpha(t_0)}(t_0)} = 0 \quad \text{with probability one.}$$

In other words, with probability one,

$$\limsup_{h\to 0} \frac{\big|Y^W_{\alpha(t_0+h)}(t_0+h) - Y^W_{\alpha(t_0)}(t_0)\big|}{|h|^{\lambda}} = \infty.$$

The Riemann–Liouville case is proved similarly.

□

5. Hausdorff dimension

Given a set A in $\mathbb{R}^n$, denote by $\dim_H A$, $\underline{\dim}_B A$ and $\overline{\dim}_B A$ the Hausdorff dimension, the lower box dimension and the upper box dimension of A respectively. It is well known that (see, e.g., [52])

$$\dim_H A \leqslant \underline{\dim}_B A \leqslant \overline{\dim}_B A, \tag{5.1}$$

and for a countable sequence of sets $\{A_i\}_{i=1}^{\infty}$,

$$\dim_H \bigcup_{i=1}^{\infty} A_i = \sup_{1\leqslant i<\infty} \{\dim_H A_i\}. \tag{5.2}$$

Given a compact interval $[a,b] \subseteq \mathbb{R}$, let $G^W_\alpha[a,b] = \big\{\big(t, Y^W_{\alpha(t)}(t)\big) : t \in [a,b]\big\}$ be the graph of the process $Y^W_{\alpha(t)}(t)$ restricted to $[a,b]$ and define $G^{RL}_\alpha[a,b]$ analogously for any $[a,b] \in \mathbb{R}^+$. We can obtain from proposition 4.1 (see, e.g., [52]) an upper bound for the box dimension:

Lemma 5.1. *Given any interval $[a,b]$ in the domain of definition, $\overline{\dim}_B G^W_\alpha[a,b] \leqslant 5/2 - m_\alpha[a,b]$ and $\overline{\dim}_B G^{RL}_\alpha[a,b] \leqslant 5/2 - m_\alpha[a,b]$ with probability one.*

Next we prove a lower bound for the Hausdorff dimension.

Lemma 5.2. *Given any interval $[a,b]$ in the domain of definition, $\dim_H G^W_\alpha[a,b] \geqslant 5/2 - M_\alpha[a,b]$ and $\dim_H G^{RL}_\alpha[a,b] \geqslant 5/2 - M_\alpha[a,b]$ with probability one.*

Proof. We use the same method as the case of multifractional Brownian motion [3]. First, if $t, t+\tau \in [a,b]$ is such that $0 < \tau < \delta$, where δ is as given by B.I. of lemma 3.1, we have for $\lambda > 1$,

$$\begin{aligned}
E\Big(\big(\big[Y^W_{\alpha(t+\tau)}(t+\tau) - Y^W_{\alpha(t)}(t)\big]^2 + \tau^2\big)^{-\lambda/2}\Big) &= \sqrt{\frac{2}{\pi}}\big(\sigma^W_{t,\tau}\big)^{-1} \int_0^\infty (u^2+\tau^2)^{-\lambda/2}\, \mathrm{e}^{-u^2/2(\sigma^W_{t,\tau})^2}\,\mathrm{d}u \\
&\leqslant \sqrt{\frac{2}{\pi}}\big(\sigma^W_{t,\tau}\big)^{-1}\left(\int_0^\tau \tau^{-\lambda}\,\mathrm{d}u + \int_\tau^\infty u^{-\lambda}\,\mathrm{d}u\right) \\
&= \sqrt{\frac{2}{\pi}}\frac{\lambda}{\lambda-1}\tau^{1-\lambda}\big(\sigma^W_{t,\tau}\big)^{-1} \\
&\leqslant \sqrt{\frac{2}{\pi}}\frac{\lambda}{\lambda-1}\big(C^W_2\big)^{-1/2}\tau^{3/2-\lambda-M_\alpha[a,b]}.
\end{aligned}$$

Given $c, d \in [a,b]$ such that $0 < d-c < \delta$, let μ be a measure defined on the graph of $Y^W_{\alpha(t)}(t)$, $t \in [c,d]$ such that

$$\mu(A) = \text{Lebesgue measure}\,\big(\big\{t : \big(t, Y^W_{\alpha(t)}(t)\big) \in A\big\}\big).$$

Let $C = \sqrt{2/\pi}[\lambda/(\lambda-1)]\big(C_2^W\big)^{-1/2}$. Then

$$
\begin{aligned}
&E\left(\int_{G_\alpha^W[c,d]}\int_{G_\alpha^W[c,d]} |x-y|^{-\lambda}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right)\\
&\qquad = \int_c^d\int_c^d E\Big(\big(\big[Y_{\alpha(t)}^W(t)-Y_{\alpha(s)}^W(s)\big]^2+(t-s)^2\big)^{-\lambda/2}\Big)\,\mathrm{d}t\,\mathrm{d}s\\
&\qquad \leqslant C\int_c^d\int_c^d |t-s|^{3/2-\lambda-M_\alpha[a,b]}\,\mathrm{d}t\,\mathrm{d}s,
\end{aligned}
$$

which is finite when $\lambda < 5/2 - M_\alpha[a,b]$. Therefore, when $1 < \lambda < 5/2 - M_\alpha[a,b]$,

$$\iint |x-y|^{-\lambda}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) < \infty \quad \text{with probability one.}$$

Therefore, by a standard result about Hausdorff dimension (see, e.g., [52]), $\dim_H G_\alpha^W[c,d] \geqslant 5/2 - M_\alpha[a,b]$. Finally by splitting the interval $[a,b]$ into finitely many sub-intervals of length less than δ and using (5.2), we conclude that $\dim_H G_\alpha^W[a,b] \geqslant 5/2 - M_\alpha[a,b]$. The Riemann–Liouville case is similar. □

Now we can conclude that

Proposition 5.3. *Given any interval* $[a,b]$ *in the domain of definition, we have with probability one,* $\dim_H G_\alpha^W[a,b] = \underline{\dim}_B G_\alpha^W[a,b] = \overline{\dim}_B G_\alpha^W[a,b] = 5/2 - m_\alpha[a,b]$ *and* $\dim_H G_\alpha^{RL}[a,b] = \underline{\dim}_B G_\alpha^{RL}[a,b] = \overline{\dim}_B G_\alpha^{RL}[a,b] = 5/2 - m_\alpha[a,b]$.

Proof. In view of (5.1) and lemma 5.1, we need only to show that with probability one,

$$\dim_H G_\alpha^W[a,b] \geqslant 5/2 - m_\alpha[a,b] \qquad \text{and} \qquad \dim_H G_\alpha^{RL}[a,b] \geqslant 5/2 - m_\alpha[a,b].$$

Let $t_0 \in [a,b]$ be such that $m_\alpha[a,b] = \alpha(t_0)$. Let $u_0, u_1, u_2, \ldots$ be a sequence of points such that

$$t_0 - u_i = (t_0 - a)/2^i.$$

Then by (5.2),

$$\dim_H G^W[a,b] = \max\Big\{\sup_{1\leqslant i<\infty}\big\{\dim_H G_\alpha^W[u_{i-1},u_i]\big\}, \dim_H G_\alpha^W[t_0,b]\Big\}.$$

By lemma 5.2,

$$\dim_H G^W[u_{i-1},u_i] \geqslant 5/2 - M_\alpha[u_{i-1},u_i] \qquad \text{for all} \quad i.$$

Therefore,

$$\dim_H G_\alpha^W[a,b] \geqslant 5/2 - M_\alpha[u_{i-1},u_i] \qquad \text{for all} \quad i.$$

But by the continuity of α at t_0,

$$\lim_{i\to\infty} M_\alpha[u_{i-1},u_i] = \alpha(t_0) = m_\alpha[a,b].$$

This gives us the assertion $\dim_H G^W[a,b] \geqslant 5/2 - m_\alpha[a,b]$. The proof for the Riemann–Liouville case is the same. □

6. Local asymptotic self-similarity

The multifractional Ornstein–Uhlenbeck processes $Y^{W}_{\alpha(t)}(t)$ and $Y^{RL}_{\alpha(t)}(t)$ are not self-similar processes. However, as the multifractional Brownian motion is locally self-similar, we might expect that the same thing holds for multifractional Ornstein–Uhlenbeck processes. As defined in [4], a Gaussian process $\{X(t)\}$ is called locally asymptotically self-similar with parameter H at a point t_0 if the limit process

$$\left\{\lim_{h\to 0^+}\frac{X(t_0+hu)-X(t_0)}{h^H}, u\in\mathbb{R}\right\}$$

exists and is nontrivial for every t_0. Many time series extracted from nature exhibit local asymptotic self-similarity and tools have been developed to analyse this property [53].

We first recall that the fractional Brownian motion with Hurst parameter H is uniquely characterized as the centred Gaussian process with covariance

$$E(B_H(t)B_H(s)) = -\frac{1}{2\Gamma(2H+1)\cos\pi(H+1/2)}(|t|^{2H}+|s|^{2H}-|t-s|^{2H}).$$

Our main result in this section is that the multifractional Ornstein–Uhlenbeck processes are locally asymptotically self-similar. Moreover,

Proposition 6.1. *For any t_0, the stochastic process*

$$\left\{\lim_{h\to 0^+}\frac{Y^{W}_{\alpha(t_0+hu)}(t_0+hu)-Y^{W}_{\alpha(t_0)}(t_0)}{h^{\alpha(t_0)-1/2}}, u\in\mathbb{R}\right\}$$

is a fractional Brownian motion of Hurst parameter $\alpha(t_0)-1/2$. Similar result holds for the Riemann–Liouville multifractional Ornstein–Uhlenbeck process.

Proof. It is obvious that for any t_0 and u, the random variable

$$\lim_{h\to 0^+}\frac{Y^{W}_{\alpha(t_0+hu)}(t_0+hu)-Y^{W}_{\alpha(t_0)}(t_0)}{h^{\alpha(t_0)-1/2}},$$

if the limit exists, is a normal distribution with mean 0. Now for $h>0$,

$$E\big(\big[Y^{W}_{\alpha(t_0+hu)}(t_0+hu)-Y^{W}_{\alpha(t_0)}(t_0)\big]\big[Y^{W}_{\alpha(t_0+hv)}(t_0+hv)-Y^{W}_{\alpha(t_0)}(t_0)\big]\big)$$
$$=\frac{1}{2}\big(\big(\sigma^{W}_{t_0,hu}\big)^2+\big(\sigma^{W}_{t_0,hv}\big)^2-\big(\sigma^{W}_{t_0+hu,hv-hu}\big)^2\big).$$

From B.II of lemma 3.1, we obtain immediately

$$\lim_{h\to 0^+}\frac{\big(\sigma^{W}_{t_0,h\tau}\big)^2}{h^{2\alpha(t_0)-1}} = -\frac{|\tau|^{2\alpha(t_0)-1}}{\Gamma(2\alpha(t_0))\cos(\pi\alpha(t_0))}$$

for any τ. On the other hand,

$$\lim_{h\to 0^+}\frac{\big(\sigma^{W}_{t_0+hu,hv-hu}\big)^2}{h^{2\alpha(t_0)-1}} = \lim_{h\to 0^+}h^{2\alpha(t_0+hu)-2\alpha(t_0)}\lim_{h\to 0^+}\left(-\frac{|u-v|^{2\alpha(t_0+hu)-1}}{\Gamma(2\alpha(t_0+hu))\cos(\pi\alpha(t_0+hu))}\right).$$

The second limit can be easily evaluated using the continuity of α. For the first limit,

$$\lim_{h\to 0^+}h^{2\alpha(t_0+hu)-2\alpha(t_0)} = \exp\big(\lim_{h\to 0^+}(2\alpha(t_0+hu)-2\alpha(t_0))\log h\big).$$

By Hölder condition of α,

$$|(2\alpha(t_0+hu)-2\alpha(t_0))\log h| \leqslant 2K|u|^{\beta}|h^{\beta}\log h|\to 0 \qquad \text{as} \quad h\to 0.$$

Therefore,

$$\lim_{h\to 0^+} \frac{\left(\sigma^W_{t_0+hu,hv-hu}\right)^2}{h^{2\alpha(t_0)-1}} = -\frac{|u-v|^{2\alpha(t_0)-1}}{\Gamma(2\alpha(t_0))\cos(\pi\alpha(t_0))}.$$

These imply that

$$E\left(\lim_{h\to 0^+}\left(\frac{Y^W_{\alpha(t_0+hu)}(t_0+hu)-Y^W_{\alpha(t_0)}(t_0)}{h^{\alpha(t_0)-1/2}}\right)\left(\frac{Y^W_{\alpha(t_0+hv)}(t_0+hv)-Y^W_{\alpha(t_0)}(t_0)}{h^{\alpha(t_0)-1/2}}\right)\right)$$
$$= -\frac{1}{2\Gamma(2\alpha(t_0))\cos(\pi\alpha(t_0))}(|u|^{2\alpha(t_0)-1}+|v|^{2\alpha(t_0)-1}-|u-v|^{2\alpha(t_0)-1}),$$

which is the assertion for the Weyl process. The Riemann–Liouville case is proved in the same way. □

7. Short range dependence property

Now we consider the strength or extent of the correlations of the Weyl and Riemann Liouville multifractional Ornstein–Uhlenbeck processes. In other words, we want to know whether these processes have long-range dependence (LRD or long memory) or short-range dependence (SRD or short memory). Given a Gaussian process $X(t)$ with covariance $C(s,t) = \mathrm{Cov}(X(s)X(t))$, let $R(s,t)$ be its correlation function, i.e.

$$R(s,t) = \frac{C(X(s)X(t))}{\sqrt{C(t,t)C(s,s)}}.$$

As in [54], we say that $X(t)$ is LRD if the integral

$$\int_0^\infty |R(t,t+\tau)|\,\mathrm{d}\tau = \infty$$

and it is SRD if the integral is finite.

First we need some estimates.

Lemma 7.1. *A. There exist constants C^W_3 such that for all t,*

$$\frac{1}{E\left(\left[Y^W_{\alpha(t)}(t)\right]^2\right)} \leqslant C^W_3.$$

B. Given $t\in\mathbb{R}^+$, there exists a constant $C^{RL}_{3,t}$ depending on t such that

$$\frac{1}{E\left(\left[Y^{RL}_{\alpha(t+\tau)}(t+\tau)\right]^2\right)} \leqslant C^{RL}_{3,t}$$

for all $\tau \geqslant 0$.

Proof. From (2.6), we have

$$\frac{1}{E\left(\left[Y^W_{\alpha(t)}(t)\right]^2\right)} = \frac{(2\omega)^{2\alpha(t)-1}\Gamma(\alpha(t))^2}{\Gamma(2\alpha(t)-1)} = \frac{(2\omega)^{2\alpha(t)-1}\Gamma(\alpha(t))^2(2\alpha(t)-1)}{\Gamma(2\alpha(t))}.$$

Since $\alpha(t)\in(1/2,3/2), 0<2\alpha(t)-1<2$. Moreover, $z\mapsto(2\omega)^{2z-1}, z\mapsto\Gamma(z)$ and $z\mapsto 1/\Gamma(2z)$ are continuous functions for $z\in[1/2,3/2]$. Therefore $1/E\left(\left[Y^W_{\alpha(t)}(t)\right]^2\right)$ is bounded above by a constant C^W_3 independent of t.

For the Riemann–Liouville case, we have

$$E\left(\left[Y^{RL}_{\alpha(t+\tau)}(t+\tau)\right]^2\right) = \frac{1}{\Gamma(\alpha(t+\tau))^2}\int_0^{t+\tau} u^{2\alpha(t+\tau)-2}\,\mathrm{e}^{-2\omega u}\,\mathrm{d}u.$$

Now for $\tau \geqslant 0$,

$$\int_0^{t+\tau} u^{2\alpha(t+\tau)-2}\,\mathrm{e}^{-2\omega u}\,\mathrm{d}u \geqslant \int_0^{t} u^{2\alpha(t+\tau)-2}\,\mathrm{e}^{-2\omega u}\,\mathrm{d}u.$$

Since $1/2 \leqslant \alpha(t+\tau) \leqslant 3/2$ for all τ, if $t \leqslant 1$,

$$\int_0^{t} u^{2\alpha(t+\tau)-2}\,\mathrm{e}^{-2\omega u}\,\mathrm{d}u \geqslant \mathrm{e}^{-2\omega t}\int_0^t u\,\mathrm{d}u = \frac{t^2}{2}\,\mathrm{e}^{-2\omega t}.$$

If $t > 1$, then

$$\int_0^{t} u^{2\alpha(t+\tau)-2}\,\mathrm{e}^{-2\omega u}\,\mathrm{d}u \geqslant \mathrm{e}^{-2\omega}\int_0^1 u\,\mathrm{d}u = \frac{1}{2}\,\mathrm{e}^{-2\omega}.$$

In any case,

$$\frac{1}{E\big(\big[Y^{RL}_{\alpha(t+\tau)}(t+\tau)\big]^2\big)} \leqslant M\max\left\{\frac{2\,\mathrm{e}^{2\omega t}}{t^2}, 2\,\mathrm{e}^{2\omega}\right\},$$

where M is the maximum value of $\Gamma(z)^2$ for $z \in [1/2, 3/2]$. □

Now we can prove the SRD property of $Y^W_{\alpha(t)}(t)$ and $Y^{RL}_{\alpha(t)}(t)$.

Proposition 7.2. *Both the Weyl and Riemann–Liouville multifractional Ornstein–Uhlenbeck processes have short memory.*

Proof. For the Weyl multifractional Ornstein–Uhlenbeck process, the correlation function is given by

$$R^W_\alpha(t, t+\tau) = \frac{E\big(Y^W_{\alpha(t)}(t)Y^W_{\alpha(t+\tau)}(t+\tau)\big)}{\sqrt{E\big(\big[Y^W_{\alpha(t)}(t)\big]^2\big)E\big(\big[Y^W_{\alpha(t+\tau)}(t+\tau)\big]^2\big)}}.$$

By (2.7) and lemma 7.1, we have

$$0 \leqslant R^W_\alpha(t, t+\tau) \leqslant C^W_3 M^2\mathrm{e}^{-\omega\tau}\int_0^\infty u^{\alpha(t)-1}(u+\tau)^{\alpha(t+\tau)-1}\,\mathrm{e}^{-2\omega u}\,\mathrm{d}u,$$

where M is the maximum value of $1/\Gamma(z)$ for $z \in [1/2, 3/2]$. Therefore,

$$\begin{aligned}\int_0^\infty \big|R^W_\alpha(t, t+\tau)\big|\,\mathrm{d}\tau &\leqslant C^W_3 M^2\int_0^\infty \mathrm{e}^{-\omega\tau}\left(\int_0^\infty u^{\alpha(t)-1}(u+\tau)^{\alpha(t+\tau)-1}\,\mathrm{e}^{-2\omega u}\,\mathrm{d}u\right)\mathrm{d}\tau\\ &= C^W_3 M^2\int_0^\infty u^{\alpha(t)-1}\,\mathrm{e}^{-\omega u}\left(\int_0^\infty (u+\tau)^{\alpha(t+\tau)-1}\,\mathrm{e}^{-\omega(u+\tau)}\,\mathrm{d}\tau\right)\mathrm{d}u\\ &\leqslant C^W_3 M^2\left(\int_0^\infty u^{\alpha(t)-1}\,\mathrm{e}^{-\omega u}\,\mathrm{d}u\right)\left(\int_0^\infty \tau^{\alpha(t+\tau)-1}\,\mathrm{e}^{-\omega\tau}\,\mathrm{d}\tau\right).\end{aligned}$$

Now using again $1/2 \leqslant \alpha(s) \leqslant 3/2$ for all s, we have

$$\begin{aligned}\int_0^\infty \tau^{\alpha(t+\tau)-1}\,\mathrm{e}^{-\omega\tau}\,\mathrm{d}\tau &\leqslant \int_0^1 \tau^{-1/2}\mathrm{d}\tau + \int_1^\infty \tau^{1/2}\,\mathrm{e}^{-\omega\tau}\,\mathrm{d}\tau\\ &\leqslant 2 + \int_0^\infty \tau^{1/2}\,\mathrm{e}^{-\omega\tau}\,\mathrm{d}\tau\\ &= 2 + \omega^{-3/2}\Gamma(3/2).\end{aligned}$$

Similarly,

$$\int_0^\infty u^{\alpha(t)-1}\,\mathrm{e}^{-\omega u}\,\mathrm{d}u \leqslant 2 + \omega^{-3/2}\Gamma(3/2).$$

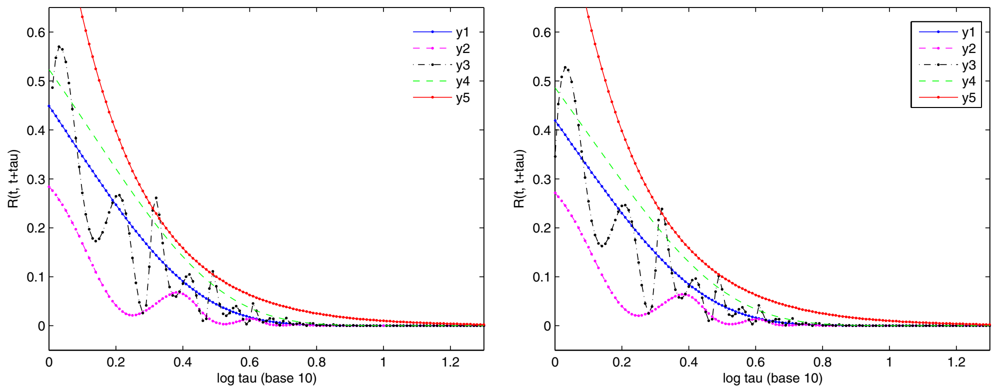

Figure 3. The correlation function $R(t, t+\tau)$ with $t = 1, \omega = 1$ for the Weyl multifractional Ornstein–Uhlenbeck process $Y^W_{\alpha(t)}(t)$ (left) and the Riemann–Liouville multifractional Ornstein–Uhlenbeck process $Y^{RL}_{\alpha(t)}(t)$ (right). For $y1, \alpha(t) = 0.5 + 0.8\,\mathrm{e}^{-0.1t}$; for $y2, \alpha(t) = 0.8 + 0.25 \sin(4t)$; for $y3, \alpha(t) = 1 + 125\{t\}(\{t\} - 0.25)(\{t\} - 0.5)(\{t\} - 0.75)(\{t\} - 1)$ where $\{t\} = t - [t]$; for $y4, \alpha(t) = 1.3$. $y5$ is the graph of the function $y = 1/t^2$.

These imply our assertion that

$$\int_0^\infty \left|R^W_\alpha(t, t+\tau)\right| \mathrm{d}\tau \text{ is finite.}$$

For the Riemann–Liouville case, we rewrite (2.10) as

$$E\left(Y^{RL}_{\alpha(t)}(t)Y^{RL}_{\alpha(t+\tau)}(t+\tau)\right) = \frac{\mathrm{e}^{-\omega\tau}}{\Gamma(\alpha(t))\Gamma(\alpha(t+\tau))}\int_0^t u^{\alpha(t)-1}(u+\tau)^{\alpha(t+\tau)-1}\,\mathrm{e}^{-2\omega u}\,\mathrm{d}u.$$

It is then obvious that

$$0 \leqslant E\left(Y^{RL}_{\alpha(t)}(t)Y^{RL}_{\alpha(t+\tau)}(t+\tau)\right) \leqslant E\left(Y^{W}_{\alpha(t)}(t)Y^{W}_{\alpha(t+\tau)}(t+\tau)\right).$$

Therefore, the SRD property of the Riemann–Liouville process follows from the SRD property of the Weyl process and lemma 7.1. □

In contrast to multifractional Ornstein–Uhlenbeck processes, multifractional Brownian motions, which are their massless limits, have long-range dependence. Consider first the Riemann–Liouville multifractional Brownian motion. Using the formulae in (2.11) of the variance and covariance, and the fact that ${}_2F_1(a, b; c; z) \to 1$ as $z \to 0$, we find that for fixed t, the corresponding correlation function $R(t, t+\tau)$ behaves like $\tau^{-1/2}$ as $\tau \to \infty$. Therefore, the integral

$$\int_0^\infty |R(t, t+\tau)|\,\mathrm{d}\tau$$

is divergent and the Riemann–Liouville multifractional Brownian motion is LRD.

For the multifractional Brownian motion (2.9) with covariance given by (2.8), since as $\tau \to \infty$,

$$(t+\tau)^{H(t+\tau)+H(t)} = \tau^{H(t+\tau)+H(t)}(1 + (H(t+\tau) + H(t))t\tau^{-1} + O(\tau^{-2})),$$

we find that the corresponding correlation function $R(t, t+\tau)$ behaves like $\tau^{-H(t+\tau)}$ if $0 < H(t) + H(t+\tau) \leqslant 1$, and behaves like $\tau^{-(1-H(t))}$ if $1 < H(t) + H(t+\tau) < 2$.

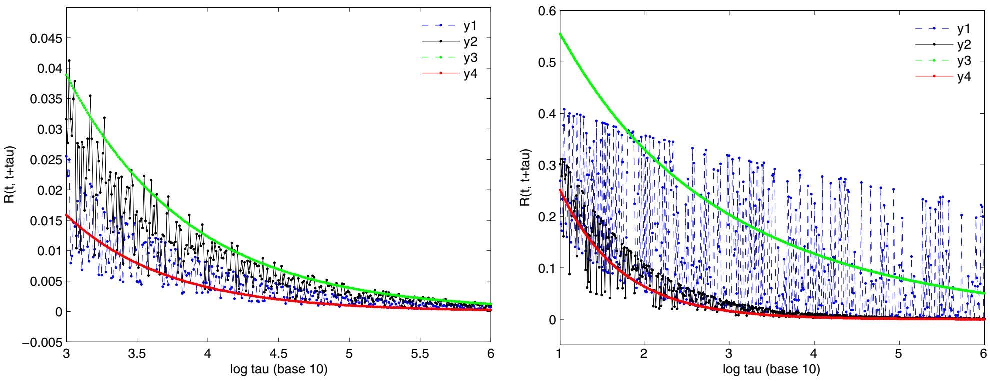

Figure 4. The correlation function $R(t, t+\tau)$ with $t = 1$ for the Riemann–Liouville multifractional Brownian motion (left) and the multifractional Brownian motion (2.9) (right). For $y1$, $\alpha(t) = 0.8 + 0.25\sin(4t)$; for $y2$, $\alpha(t) = 1 + 125\{t\}(\{t\} - 0.25)(\{t\} - 0.5)(\{t\} - 0.75)(\{t\} - 1)$ where $\{t\} = t - [t]$; for $y3$, $\alpha(t) = 1.3$. $y4$ is the graph of the function $y = 1/t^{0.6}$.

Therefore, the integral

$$\int_0^\infty |R(t, t+\tau)|\,\mathrm{d}\tau$$

is also divergent and the multifractional Brownian motion is LRD.

In figures 3 and 4, we show the graph of the correlation function $R(t, t+\tau)$ for the Weyl and Riemann–Liouville multifractional Ornstein–Uhlenbeck processes as well as the Riemann–Liouville multifractional Brownian motion and the multifractional Brownian motion (2.9), for some particular choices of $\alpha(t)$.

8. Concluding remarks

This paper introduces a new multifractional Gaussian process with short-range correlation. Two versions of this process, namely the Weyl multifractional Ornstein–Uhlenbeck process and the Riemann–Liouville multifractional Ornstein–Uhlenbeck process were discussed. Various basic properties of the multifractional Ornstein–Uhlenbeck processes such as the short-range dependence, local asymptotic self-similarity and local Hausdorff dimension are studied. We also compare the properties of the multifractional Ornstein–Uhlenbeck process with that of multifractional Brownian motion. In the case of the Riemann–Liouville multifractional Ornstein–Uhlenbeck process, its 'massless' limit (or $\omega \to 0$ limit) gives exactly the Riemann–Liouville version of multifractional Brownian motion. For the Weyl multifractional Ornstein–Uhlenbeck process, the 'massless' limit of its reduced process is the multifractional Brownian motion (of moving average definition).

The remark concerning the multifractality of the multifractional Brownian motion applies to the multifractional Ornstein–Uhlenbeck process. That is, the multifractional process is truly multifractal if the Hölder exponent is a random function, otherwise it is a multiscaling process [7]. However, there are many phenomena that are multiscaling instead of multifractal (see, e.g., [55]). The generalization of the multifractional Ornstein–Uhlenbeck process to the multifractional Ornstein–Uhlenbeck field and sheet can be carried out in a similar way as in the multifractional Brownian motion [6, 12, 13]. It will be interesting to study the fractional Langevin equation with variable order as the equation that governs the multifractional process. There are some works along this direction which consider fractional differential operators and fractional differential equations with variable order [56–59].

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Appendix A. Verification of formula (2.8)

In this appendix, we want to show that when $\omega \to 0$, the limit of the process $X^W_{\alpha(t)}(t) = Y^W_{\alpha(t)}(t) - Y^W_{\alpha(t)}(0)$ is the multifractional Brownian motion $B_{H(t)}(t)$ (2.9) if we replace $\alpha(t)$ by $H(t) + (1/2)$. For this purpose, it is enough to verify (2.8). We have

$$E\left(\left[X^W_{\alpha(t)}(t)X^W_{\alpha(s)}(s)\right]\right) = E\left(\left[Y^W_{\alpha(t)}(t)Y^W_{\alpha(s)}(s)\right]\right) + E\left(\left[Y^W_{\alpha(t)}(0)Y^W_{\alpha(s)}(0)\right]\right) - E\left(\left[Y^W_{\alpha(t)}(t)Y^W_{\alpha(s)}(0)\right]\right) - E\left(\left[Y^W_{\alpha(s)}(s)Y^W_{\alpha(t)}(0)\right]\right).$$

By (2.7), we have for $t > s$,

$$E\left(\left[Y^W_{\alpha_1}(t)Y^W_{\alpha_2}(s)\right]\right) = \frac{e^{-\omega(t-s)}(t-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1)}\Psi(\alpha_2, \alpha_1+\alpha_2, 2\omega(t-s)),$$

whereas when $t = s$, we have

$$E\left(\left[Y^W_{\alpha_1}(t)Y^W_{\alpha_2}(t)\right]\right) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^\infty u^{\alpha_1+\alpha_2-2}\, e^{-2\omega u}\, du = \frac{(2\omega)^{1-\alpha_1-\alpha_2}\Gamma(\alpha_1+\alpha_2-1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}.$$

Using (see, e.g., 9.210 of [41])

$$\Psi(\beta, \gamma; z) = \frac{\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)}\sum_{m=0}^\infty \frac{(\beta)_m}{(\gamma)_m}\frac{z^m}{m!} + \frac{\Gamma(\gamma-1)}{\Gamma(\beta)}z^{1-\gamma}\sum_{m=0}^\infty \frac{(\beta-\gamma+1)_m}{(2-\gamma)_m}\frac{z^m}{m!},$$

where $(\beta)_m = \beta(\beta+1)\ldots(\beta+m-1)$ is the Pochhammer symbol, we have for $\gamma < 3$,

$$\Psi(\beta, \gamma; z) = \frac{\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)} + \frac{\Gamma(\gamma-1)}{\Gamma(\beta)}z^{1-\gamma}\left(1 + \frac{\beta-\gamma+1}{2-\gamma}z\right) + O(z^{\min\{1,3-\gamma\}}).$$

To find the $\omega \to 0$ limit of $E\left(\left[X^W_{\alpha(t)}(t)X^W_{\alpha(s)}(s)\right]\right)$, we need to divide subcases. For notational convenience, in the following, we write $\alpha(t) = \alpha_t$, $\alpha(s) = \alpha_s$ and $\Delta_{t,s} = t - s$.

A.1. Case I. $s < t < 0$

In this case,

$$\begin{aligned} E\left(\left[X^W_{\alpha(t)}(t)X^W_{\alpha(s)}(s)\right]\right) &= \frac{e^{-\omega\Delta_{t,s}}\Delta_{t,s}^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\Psi(\alpha_s, \alpha_t+\alpha_s, 2\omega\Delta_{t,s}) \\ &\quad + \frac{(2\omega)^{1-\alpha_t-\alpha_s}\Gamma(\alpha_t+\alpha_s-1)}{\Gamma(\alpha_t)\Gamma(\alpha_s)} \\ &\quad - \frac{e^{\omega t}(-t)^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_s)}\Psi(\alpha_t, \alpha_t+\alpha_s, -2\omega t) \\ &\quad - \frac{e^{\omega s}(-s)^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\Psi(\alpha_s, \alpha_t+\alpha_s, -2\omega s) \\ &= T_1 + T_2 + O(\omega^{\min\{1,3-\gamma\}}), \end{aligned}$$

where

$$T_1 = \frac{\mathrm{e}^{-\omega\Delta_{t,s}}\Delta_{t,s}^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(1-\alpha_t-\alpha_s)}{\Gamma(1-\alpha_t)} - \frac{\mathrm{e}^{\omega t}(-t)^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_s)}\frac{\Gamma(1-\alpha_t-\alpha_s)}{\Gamma(1-\alpha_s)}$$
$$- \frac{\mathrm{e}^{\omega s}(-s)^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(1-\alpha_t-\alpha_s)}{\Gamma(1-\alpha_t)}$$
$$= \frac{\Gamma(1-\alpha_t-\alpha_s)}{\pi}(\sin(\pi\alpha_t)|\Delta_{t,s}|^{\alpha_t+\alpha_s-1} - \sin(\pi\alpha_s)|t|^{\alpha_t+\alpha_s-1}$$
$$- \sin(\pi\alpha_t)|s|^{\alpha_t+\alpha_s-1}) + O(\omega),$$

$$T_2 = \frac{\mathrm{e}^{-\omega\Delta_{t,s}}\Delta_{t,s}^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(\alpha_s+\alpha_t-1)}{\Gamma(\alpha_s)}[2\omega\Delta_{t,s}]^{1-\alpha_s-\alpha_t}\left(1+\frac{2(1-\alpha_t)}{2-\alpha_s-\alpha_t}\omega\Delta_{t,s}\right)$$
$$+ \frac{(2\omega)^{1-\alpha_t-\alpha_s}\Gamma(\alpha_t+\alpha_s-1)}{\Gamma(\alpha_t)\Gamma(\alpha_s)}$$
$$- \frac{\mathrm{e}^{\omega t}(-t)^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_s)}\frac{\Gamma(\alpha_s+\alpha_t-1)}{\Gamma(\alpha_t)}[-2\omega t]^{1-\alpha_s-\alpha_t}\left(1+\frac{2(1-\alpha_s)}{2-\alpha_s-\alpha_t}\omega(-t)\right)$$
$$- \frac{\mathrm{e}^{\omega s}(-s)^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(\alpha_s+\alpha_t-1)}{\Gamma(\alpha_s)}[-2\omega s]^{1-\alpha_s-\alpha_t}\left(1+\frac{2(1-\alpha_t)}{2-\alpha_s-\alpha_t}\omega(-s)\right).$$

By using the expansion $\mathrm{e}^x = 1 + x + O(x^2)$, we have

$$T_2 = \frac{\Gamma(\alpha_s+\alpha_t-1)}{\Gamma(\alpha_s)\Gamma(\alpha_t)}(2\omega)^{1-\alpha_s-\alpha_t}\left(\frac{\alpha_s-\alpha_t}{2-\alpha_s-\alpha_t}\omega\Delta_{t,s} - \frac{\alpha_t-\alpha_s}{2-\alpha_s-\alpha_t}\omega(-t)\right.$$
$$\left.- \frac{\alpha_s-\alpha_t}{2-\alpha_s-\alpha_t}\omega(-s)\right) + O(\omega^{3-\alpha_s-\alpha_t}) = O(\omega^{3-\alpha_s-\alpha_t}).$$

Therefore, as $\omega \to 0$,

$$E\left(\left[X_{\alpha_t}^W(t)X_{\alpha_s}^W(s)\right]\right) \xrightarrow{\omega\to 0} \frac{\Gamma(1-\alpha_t-\alpha_s)}{\pi}(\sin(\pi\alpha_t)|\Delta_{t,s}|^{\alpha_t+\alpha_s-1}$$
$$- \sin(\pi\alpha_s)|t|^{\alpha_t+\alpha_s-1} - \sin(\pi\alpha_t)|s|^{\alpha_t+\alpha_s-1}).$$

A.2. Case II. $t < s < 0$

This case is symmetric to case I. Therefore,

$$E\left(\left[X_{\alpha_t}^W(t)X_{\alpha_s}^W(s)\right]\right) \xrightarrow{\omega\to 0} \frac{\Gamma(1-\alpha_t-\alpha_s)}{\pi}(\sin(\pi\alpha_s)|\Delta_{t,s}|^{\alpha_t+\alpha_s-1}$$
$$- \sin(\pi\alpha_t)|s|^{\alpha_t+\alpha_s-1} - \sin(\pi\alpha_s)|t|^{\alpha_t+\alpha_s-1}).$$

A.3. Case III. $s < 0 < t$

In this case,

$$E\left(\left[X_{\alpha_t}^W(t)X_{\alpha_s}^W(s)\right]\right) = \frac{\mathrm{e}^{-\omega\Delta_{t,s}}\Delta_{t,s}^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\Psi(\alpha_s,\alpha_t+\alpha_s,2\omega\Delta_{t,s}) + \frac{(2\omega)^{1-\alpha_t-\alpha_s}\Gamma(\alpha_t+\alpha_s-1)}{\Gamma(\alpha_t)\Gamma(\alpha_s)}$$
$$- \frac{\mathrm{e}^{-\omega t}t^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\Psi(\alpha_s,\alpha_t+\alpha_s,2\omega t) - \frac{\mathrm{e}^{\omega s}(-s)^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\Psi(\alpha_s,\alpha_t+\alpha_s,-2\omega s)$$
$$= T_3 + T_4 + O(\omega^{\min\{1,3-\gamma\}}),$$

where

$$
\begin{aligned}
T_3 &= \frac{e^{-\omega\Delta_{t,s}}\Delta_{t,s}^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(1-\alpha_t-\alpha_s)}{\Gamma(1-\alpha_t)} - \frac{e^{-\omega t}t^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(1-\alpha_t-\alpha_s)}{\Gamma(1-\alpha_t)} \\
&\quad - \frac{e^{\omega s}(-s)^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(1-\alpha_t-\alpha_s)}{\Gamma(1-\alpha_t)} \\
&= \frac{\Gamma(1-\alpha_t-\alpha_s)}{\pi}(\sin(\pi\alpha_t)|\Delta_{t,s}|^{\alpha_t+\alpha_s-1} - \sin(\pi\alpha_t)|t|^{\alpha_t+\alpha_s-1} \\
&\quad - \sin(\pi\alpha_t)|s|^{\alpha_t+\alpha_s-1}) + O(\omega), \\
T_4 &= \frac{e^{-\omega\Delta_{t,s}}\Delta_{t,s}^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(\alpha_s+\alpha_t-1)}{\Gamma(\alpha_s)}[2\omega\Delta_{t,s}]^{1-\alpha_s-\alpha_t}\left(1+\frac{2(1-\alpha_t)}{2-\alpha_s-\alpha_t}\omega\Delta_{t,s}\right) \\
&\quad + \frac{(2\omega)^{1-\alpha_t-\alpha_s}\Gamma(\alpha_t+\alpha_s-1)}{\Gamma(\alpha_t)\Gamma(\alpha_s)} \\
&\quad - \frac{e^{-\omega t}t^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(\alpha_s+\alpha_t-1)}{\Gamma(\alpha_s)}[2\omega t]^{1-\alpha_s-\alpha_t}\left(1+\frac{2(1-\alpha_t)}{2-\alpha_s-\alpha_t}\omega t\right) \\
&\quad - \frac{e^{\omega s}(-s)^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(\alpha_s+\alpha_t-1)}{\Gamma(\alpha_s)}[-2\omega s]^{1-\alpha_s-\alpha_t}\left(1+\frac{2(1-\alpha_t)}{2-\alpha_s-\alpha_t}\omega(-s)\right) \\
&= \frac{\Gamma(\alpha_s+\alpha_t-1)}{\Gamma(\alpha_s)\Gamma(\alpha_t)}(2\omega)^{1-\alpha_s-\alpha_t}\left(\frac{\alpha_s-\alpha_t}{2-\alpha_s-\alpha_t}\omega\Delta_{t,s}\right. \\
&\quad \left. - \frac{\alpha_s-\alpha_t}{2-\alpha_s-\alpha_t}\omega t - \frac{\alpha_s-\alpha_t}{2-\alpha_s-\alpha_t}\omega(-s)\right) + O(\omega^{3-\alpha_s-\alpha_t}) = O(\omega^{3-\alpha_s-\alpha_t}).
\end{aligned}
$$

Therefore, as $\omega \to 0$,

$$
\begin{aligned}
E\big(\big[X_{\alpha_t}^W(t)X_{\alpha_s}^W(s)\big]\big) &\xrightarrow{\omega\to 0} \frac{\Gamma(1-\alpha_t-\alpha_s)}{\pi}(\sin(\pi\alpha_t)\|\Delta_{t,s}|^{\alpha_t+\alpha_s-1} \\
&\quad - \sin(\pi\alpha_t)|t|^{\alpha_t+\alpha_s-1} - \sin(\pi\alpha_t)|s|^{\alpha_t+\alpha_s-1}).
\end{aligned}
$$

A.4. Case IV. $t < 0 < s$

This case is symmetric to case III. Therefore,

$$
\begin{aligned}
E\big(\big[X_{\alpha_t}^W(t)X_{\alpha_s}^W(s)\big]\big) &\xrightarrow{\omega\to 0} \frac{\Gamma(1-\alpha_t-\alpha_s)}{\pi}(\sin(\pi\alpha_s)|\Delta_{t,s}|^{\alpha_t+\alpha_s-1} \\
&\quad - \sin(\pi\alpha_s)|s|^{\alpha_t+\alpha_s-1} - \sin(\pi\alpha_s)|t|^{\alpha_t+\alpha_s-1}).
\end{aligned}
$$

A.5. Case V. $0 < s < t$

In this case,

$$
\begin{aligned}
E\big(\big[X_{\alpha_t}^W(t)X_{\alpha_s}^W(s)\big]\big) &= \frac{e^{-\omega\Delta_{t,s}}\Delta_{t,s}^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\Psi(\alpha_s,\alpha_t+\alpha_s,2\omega\Delta_{t,s}) + \frac{(2\omega)^{1-\alpha_t-\alpha_s}\Gamma(\alpha_t+\alpha_s-1)}{\Gamma(\alpha_t)\Gamma(\alpha_s)} \\
&\quad - \frac{e^{-\omega t}t^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\Psi(\alpha_s,\alpha_t+\alpha_s,2\omega t) - \frac{e^{-\omega s}s^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_s)}\Psi(\alpha_t,\alpha_t+\alpha_s,2\omega s) \\
&= T_5 + T_6 + O(\omega^{\min\{1,3-\gamma\}}),
\end{aligned}
$$

where

$$T_5 = \frac{e^{-\omega\Delta_{t,s}}\Delta_{t,s}^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(1-\alpha_t-\alpha_s)}{\Gamma(1-\alpha_t)} - \frac{e^{-\omega t}t^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(1-\alpha_t-\alpha_s)}{\Gamma(1-\alpha_t)}$$
$$- \frac{e^{-\omega s}s^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_s)}\frac{\Gamma(1-\alpha_t-\alpha_s)}{\Gamma(1-\alpha_s)}$$
$$= \frac{\Gamma(1-\alpha_t-\alpha_s)}{\pi}(\sin(\pi\alpha_t)|\Delta_{t,s}|^{\alpha_t+\alpha_s-1} - \sin(\pi\alpha_t)|t|^{\alpha_t+\alpha_s-1}$$
$$- \sin(\pi\alpha_s)|s|^{\alpha_t+\alpha_s-1}) + O(\omega),$$

$$T_6 = \frac{e^{-\omega\Delta_{t,s}}\Delta_{t,s}^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(\alpha_s+\alpha_t-1)}{\Gamma(\alpha_s)}[2\omega\Delta_{t,s}]^{1-\alpha_s-\alpha_t}\left(1+\frac{2(1-\alpha_t)}{2-\alpha_s-\alpha_t}\omega\Delta_{t,s}\right)$$
$$+ \frac{(2\omega)^{1-\alpha_t-\alpha_s}\Gamma(\alpha_t+\alpha_s-1)}{\Gamma(\alpha_t)\Gamma(\alpha_s)}$$
$$- \frac{e^{-\omega t}t^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_t)}\frac{\Gamma(\alpha_s+\alpha_t-1)}{\Gamma(\alpha_s)}[2\omega t]^{1-\alpha_s-\alpha_t}\left(1+\frac{2(1-\alpha_t)}{2-\alpha_s-\alpha_t}\omega t\right)$$
$$- \frac{e^{-\omega s}s^{\alpha_t+\alpha_s-1}}{\Gamma(\alpha_s)}\frac{\Gamma(\alpha_s+\alpha_t-1)}{\Gamma(\alpha_t)}[2\omega s]^{1-\alpha_s-\alpha_t}\left(1+\frac{2(1-\alpha_s)}{2-\alpha_s-\alpha_t}\omega s\right)$$
$$= \frac{\Gamma(\alpha_s+\alpha_t-1)}{\Gamma(\alpha_s)\Gamma(\alpha_t)}(2\omega)^{1-\alpha_s-\alpha_t}\left(\frac{\alpha_s-\alpha_t}{2-\alpha_s-\alpha_t}\omega\Delta_{t,s} - \frac{\alpha_s-\alpha_t}{2-\alpha_s-\alpha_t}\omega t\right.$$
$$\left. - \frac{\alpha_t-\alpha_s}{2-\alpha_s-\alpha_t}\omega s\right) + O(\omega^{3-\alpha_s-\alpha_t}) = O(\omega^{3-\alpha_s-\alpha_t}).$$

Therefore, as $\omega \to 0$,

$$E\left(\left[X_{\alpha_t}^W(t)X_{\alpha_s}^W(s)\right]\right) \overset{\omega\to 0}{\longrightarrow} \frac{\Gamma(1-\alpha_t-\alpha_s)}{\pi}(\sin(\pi\alpha_t)|\Delta_{t,s}|^{\alpha_t+\alpha_s-1}$$
$$- \sin(\pi\alpha_t)|t|^{\alpha_t+\alpha_s-1} - \sin(\pi\alpha_s)|s|^{\alpha_t+\alpha_s-1}).$$

A.6. Case VI. $0 < t < s$

This case is symmetric to case V. Therefore,

$$E\left(\left[X_{\alpha_t}^W(t)X_{\alpha_s}^W(s)\right]\right) \overset{\omega\to 0}{\longrightarrow} \frac{\Gamma(1-\alpha_t-\alpha_s)}{\pi}(\sin(\pi\alpha_s)|\Delta_{t,s}|^{\alpha_t+\alpha_s-1}$$
$$- \sin(\pi\alpha_s)|s|^{\alpha_t+\alpha_s-1} - \sin(\pi\alpha_t)|t|^{\alpha_t+\alpha_s-1}).$$

Combining cases I–VI and replacing $\alpha(\cdot)$ by $H(\cdot) + (1/2)$, we obtain (2.8).

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